PSEUDO VALUATIONS ON HOOP-ALGEBRAS WITH RESPECT TO FILTERS

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Abstract:-
The concept of pseudo valuations on hoop-algebras is introduced, and then the relationship between pseudo-valuations and filers is investigated. Some conditions for a real-valued function to be a pseudo valuation are provided. Finally, we prove the fact that binary operations on hoop-algebras are uniformly continuous based on the notion of pseudo valuation.

Keyword:-*Hoop; Pseudo valuation; Pseudo-metric space*

1 INTRODUCTION

Hoops as ordered commutative residuated integral monoids satisfying a further conditions, were introduced by Bosbach [1]. The study of hoops is motivated by their occurrence both in universal algebra and algebraic logic, and hoop theory has experienced a tremendous growth in deep structure theorems over the recent years. From the structure theorem of finite basic hoops, one can obtain an elegant short proof of the completeness theorem for propositional basic logic introduced by Hajek [2]. Due to the fact that the sets of provable formulas in corresponding systems can be described by filters of the algebraic semantics from the logic point of view, so the filter theory plays an important role in the studying of hoops. Kondo [3] considered that fundamental properties of filters in hoops, and then pointed out that any positive filter of a hoop is implicative and fantastic. To extend the research to filter theory of hoops in [3], [4] introduced the notions of n-fold (positive) implicative filters. Some researchers applied hyper structure theory on hyper hoop, and studied the properties of hyper hoop-algebras [5, 6].

The notion of pseudo-valuations were introduced by Busneag [7] in a Hilbert algebra, and he also proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. By using a pseudo valuation, Ghorbani [9] introduced a congruence relation, and gave the quotient structure of BCI-algebras. Following the research of Zhan and Jun [10], Yang and Xin [11] investigated some characterizations of pseudo pre-valuations on EQ-algebras.

In the paper, the notion of pseudo valuation is applied to hoops, and some properties are obtained. We discuss the relationship between pseudo-valuations and filers, and also give some conditions for a real-valued function to be a pseudo valuation. By discussing the concept of pseudo-metrics induced by pseudo valuations, we obtain that binary operations on hoops are uniformly continuous.

2 Preliminaries

In this section, we give the basic definition and some results of hoop-algebras which are useful for subsequent discussions.

Definition 2.1 [8] *A hoop-algebra or briefly hoop is an algebra* $(H, \otimes, \rightarrow, 1)$ *of type* $(2,2,0)$ *such that: for any* $x, y, z \in H$,
(HP1) $(A, \otimes, 1)$ *is a commutative monoid;*
(HP2) $x \rightarrow x = 1$;
(HP3) $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$;
(HP4) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$.

The order relation "$\leq$" on a hoop $(H, \otimes, \rightarrow, 1)$ can be defined as: $x \leq y$ if and only if $x \rightarrow y = 1$ for any $x, y \in A$. It is easy to see that $(H, \leq)$ is a meet semilattice with $x \wedge y = x \otimes (x \rightarrow y)$ and 1 as the maximal element.

Proposition 2.2 [12, 13] *Let* $(H, \otimes, \rightarrow, 1)$ *be a hoop. Then the following assertions are valid: for any* $x, y, z \in H$,
(1) $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$;
(2) $x \otimes (x \rightarrow y) \leq y$, $x \otimes y \leq x \wedge y \leq x \rightarrow y$, $x \leq y \rightarrow x$;
(3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$;
(4) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$;
(5) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$;
(6)   *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$, $x \otimes z \leq y \otimes z$;
(7) $x \rightarrow y = x \rightarrow (x \wedge y)$, $x \rightarrow y \leq (x \wedge z) \rightarrow (y \wedge z)$, $x \rightarrow y \leq (x \otimes z) \rightarrow (y \otimes z)$.

Definition 2.3 [14] *Let* $(H, \otimes, \rightarrow, 1)$ *be a hoop and F a nonempty subset of H. F is called a filter if it satisfies: for any* $x, y \in H$, (1) $x, y \in F$ *implies* $x \otimes y \in F$; (2) $x \in F$ *and* $x \leq y$ *imply* $y \in F$.

It is shown that a nonempty subset F of a hoop H is a filter if and only if for any $x, y \in H$, (1) $1 \in F$; (2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$ [14].

3 Pseudo valuations on hoop-algebras

In the section, the notion of pseudo valuations on hoop-algebras is given, and some characterizations of pseudo valuations are shown. In the following, unless mentioned otherwise, any hoop-algebra $(H, \otimes, \rightarrow, 1)$ will often be referred to by its support set H.

Definition 3.1 Let $\phi: H \rightarrow R$ be a real-valued function, where R is the set of all real numbers. Then ϕ is called a pseudo valuation on A with respective a filter if it satisfies the following conditions: for any $x, y \in H$,
(1) $\phi(1) = 0$,
(2) $\phi(y) \leq \phi(x) + \phi(x \rightarrow y)$.

A pseudo valuation ϕ is called a valuation if $\phi(x) = 0$ implies $x = 1$.

Example 3.2 Let H = {0, a, b, 1} where $0 < a < b < 1$. Define the operations $\otimes$ and $\rightarrow$ on H as follows:

$\otimes$	0	a	b	1		$\rightarrow$	0	a	b	1
0	0	0	0	0		0	1	1	1	1
a	0	0	0	a		a	a	1	1	1
b	0	a	b	b		b	0	a	1	1
1	0	a	b	1		1	0	a	b	1

Then $(H, \otimes, \to, 1)$ is a hoop. Define $\phi: H \to R$ by $\phi(0) = 7$, $\phi(a) = 6$, $\phi(b) = 3$, $\phi(1) = 0$. Routine calculation shows that is a pseudo valuation.

Proposition 3.3 Let ϕ be a pseudo valuation on H. Then the following inequalities are valid: for any $x, y, z \in H$,
(1) $x \le y$ implies $\phi(y) \le \phi(x)$;
(2) $0 \le \phi(x)$.
(3) $\phi(x \to z) \le \phi(x \to y) + \phi(y \to z)$;
(4) $0 \le \phi(x \to y) + \phi(y \to x)$;
(5) $\phi(x \to (y \to z)) \le \phi((x \to y) \to z)$.

Proof. (1) Let $x, y \in H$ such that $x \le y$, then we get that $x \to y = 1$, and so
$\phi(y) \le \phi(x) + \phi(x \to y) = \phi(x) + \phi(1) = \phi(x) + 0 = \phi(x)$.
(2)   For any $x \in H$, then we obtain
$0 = \phi(1) \le \phi(x \to 1) + \phi(x) = \phi(x)$.
(3)   According to Proposition 2.2 (3), we have $x \to y \le (y \to z) \to (x \to z)$ for any $x, y, z \in H$. It follows from (1) and Definition 3.1 (2) that $\phi(x \to y) \ge \phi((y \to z) \to (x \to z)) \ge \phi(x \to z) - \phi(y \to z)$,
and therefore (3) holds.
(4)   Considering that ϕ is a pseudo valuation on H, we get that $\phi(y) \le \phi(x) + \phi(x \to y)$ and $\phi(x) \le \phi(y) + \phi(y \to x)$ for any $x, y \in H$, that is, $\phi(y) - \phi(x) \le \phi(x \to y)$ and $\phi(x) - \phi(y) \le \phi(y \to x)$. It follows that $0 \le \phi(x \to y) + \phi(y \to x)$.
(5)   From Proposition 2.2 (2), we obtain that $x \otimes y \le x \to y$ for any $x, y \in H$, then $(x \to y) \to z \le (x \otimes y) \to z = x \to (y \to z)$ and thus $\phi(x \to (y \to z)) \le \phi((x \to y) \to z)$.
We will next display the relationship between pseudo valuations and filters of hoops. For this purpose, we adopt the following notation.
Let ϕ a real-valued function on a hoop-algebra H. We consider the following set:
$$F_\phi := \{x \in H | \phi(x) = 0\}.$$

Theorem 3.4 Let ϕ be a pseudo valuation on H. Then the set F_ϕ is a filter of H which is called the filter induced by ϕ.

Proof. From $\phi(1) = 0$, we obtain that $1 \in F_\phi$. For any $x, x \to y \in F_\phi$, that is, $\phi(x) = 0$ and $\phi(x \to y) = 0$. we get that $\phi(y) \le \phi(x) + \phi(x \to y) = 0$, therefore $\phi(y) = 0$, and then $y \in F_\phi$. Hence F_ϕ is a filter of H.
We would like to point out that the converse of Theorem 3.4 may not be true, the following example shows that there exist a real-valued function ϕ on a hoop H such that F_ϕ is a filter of H but ϕ is not a pseudo valuation. c1
Example 3.5 Let $H = \{0, a, b, c, 1\}$ be a set with the Hasse diagram and Cayley tables as follows.
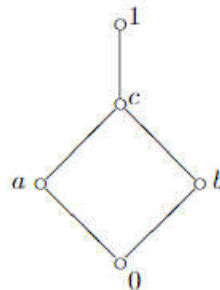
$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

According to [12], we have that $(H, \otimes, \to, 1)$ is a hoop. Define a real-valued function $\phi: H \to R$ by $\phi(0) = 3$, $\phi(c) = \phi(1) = 0$, $\phi(a) = \phi(b) = 2$. Then $F_\phi = \{c, 1\}$ is a filter of H, but ϕ is not a pseudo valuation since
$\varphi(0) = 3 \nleq \varphi(a) + \varphi(a \to 0) = 4$.
Proposition 3.6 Let α be a positive element of R and F a filter of H. Define
$$\varphi^F(x) = \begin{cases} 0, & x \in F, \\ \alpha, & x \notin F, \end{cases}$$
then ϕ^F is a pseudo pre-valuation on H. Moreover, $F_{\phi^F} = F$.
Proof. It is easy to see that ϕ^F is a pseudo pre-valuation on H and $F_{\phi^F} = \{x \in H | \phi^F(x) = 0\} = \{x \in H | x \in F\} = F$.
In the following, some conditions for a real-valued function to be a pseudo pre-valuation are present

Proposition 3.7 Let ϕ be a real-valued function on H with $\phi(1) = 0$. If $\phi(x \to (x \to z)) \leq \phi(x \to y) + \phi(x \to (y \to z))$ for any $x,y,z \in$ H, then ϕ is a pseudo valuation on H.

Proof. For any $x,y \in$ H, since $1 \to x = x$, then $\phi(x) + \phi(x \to y) = \phi(1 \to x) + \phi(1 \to (x \to y)) \geq \phi(1 \to (1 \to y)) = \phi(y)$. And so ϕ is a pseudo valuation.

Proposition 3.8 Let ϕ be a real-valued function on H with $\phi(1) = 0$. If $\phi(x) \leq \phi(z \to ((x \to y) \to x)) + \phi(z)$ for any $x,y,z \in$ H, then ϕ is a pseudo valuation on H.

Proof. For any $x,y \in$ H, it follows from $y \leq 1 \to y$ that $x \to y \leq x \to (1 \to y)$. To show ϕ is a pseudo valuation, we need the result: if $x \leq y$, then $1 = x \to (1 \to y) = x \to ((y \to y) \to y)$, and so $\phi(y) \leq \phi(x \to ((y \to y) \to y)) + \phi(x) = \phi(x)$, that is, $x \leq y$ implies that $\phi(y) \leq \phi(x)$. Hence $\phi(x) + \phi(x \to y) \geq \phi(x) + \phi(x \to (1 \to y)) = \phi(x) + \phi(x \to ((y \to 1) \to y)) \geq \phi(y)$ for any $x,y \in$ H,
and so ϕ is a pseudo valuation.

Theorem 3.9 Let ϕ be a real-valued function on H with $\phi(1) = 0$. Then ϕ is a pseudo valuation if and only if $x \leq y \to z$ implies $\phi(z) \leq \phi(x) + \phi(y)$ for any $x,y,z \in$ H.

Proof. Suppose that ϕ is a pseudo valuation. For any $x,y,z \in$ H, if $x \leq y \to z$, then $\phi(y \to z) \leq \phi(x)$, and so $\phi(z) \leq \phi(y) + \phi(y \to z) \leq \phi(x) + \phi(y)$.
Conversely, since $x \to y \leq x \to y$ for any $x,y \in$ H, then $\phi(y) \leq \phi(x \to y) + \phi(x)$, and therefore ϕ is a pseudo valuation.

Proposition 3.10 Let ϕ be a pseudo valuation on H. Then for any $x,y,w,v \in$ H, we have:
(1) $\max\{\phi(x) - \phi(y), \phi(y) - \phi(x)\} \leq \phi(x \otimes y) \leq \phi(x) + \phi(y)$;
(2) $\max\{\phi(x) - \phi(y), \phi(y) - \phi(x)\} \leq \phi(x \wedge y) \leq \phi(x) + \phi(y)$;
(3) $\phi((x \wedge w) \to (y \wedge v)) \leq \phi(x \to y) + \phi(w \to v)$;
(4) $\phi((x \to w) \to (y \to v)) \leq \phi(y \to x) + \phi(w \to v)$,

Proof. (1) Since $x \otimes y \leq x \otimes y$, then $x \leq y \to (x \otimes y)$, and so $\phi(x \otimes y) \leq \phi(x) + \phi(y)$ by Theorem 3.9. As for the left inequality, from $x \otimes y \leq x \to y$ and $y \otimes x \leq y \to x$, we obtain that $\phi(y) \leq \phi(x \otimes y) + \phi(x)$ and $\phi(x) \leq \phi(x \otimes y) + \phi(y)$, that is, $\phi(y) - \phi(x) \leq \phi(x \otimes y)$ and $\phi(x) - \phi(y) \leq \phi(x \otimes y)$, hence (1) holds. (2) The proof is similar to (1).
(3)   For any $x,y,w,v \in$ H, we get that $x \to y \leq (x \wedge w) \to (y \wedge w)$ and $w \to v \leq (y \wedge w) \to (y \wedge v)$ by Proposition 2.2 (7). According to Proposition 3.3, we obtain that $\phi(x \to y) + \phi(w \to v) \geq \phi((x \wedge w) \to (y \wedge w)) + \phi((y \wedge w) \to (y \wedge v)) \geq \phi((x \wedge w) \to (y \wedge v))$.
(4)   According to Proposition 2.2 (3) and Proposition 3.3, for any $x,y,w,v \in$ H, we have that $\phi((x \to w) \to (y \to v)) \leq \phi((x \to w) \to (y \to w)) + \phi((y \to w) \to (y \to v)) \leq \phi(y \to x) + \phi(w \to v)$.
Let (M,d) be an ordered pair, where M is a nonempty set and $d : M \times M \to R$ is a positive function.
If d satisfies the following conditions: for any $x,y,z \in$ M,
(1) $d(x,x) = 0$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,z) \leq d(x,y) + d(y,z)$, then (M,d) is called a pseudo-metric space. Moreover, if $d(x,y) = 0$ implies $x = y$, then (M,d) is called a metric space.

Theorem 3.11 Let ϕ be a pseudo valuation on H. Define a real-valued function d_ϕ by $d_\phi(x,y) = \phi(x \to y) + \phi(y \to x)$ for any $x,y \in$ H, then (H, d_ϕ) is a pseudo-metric space. And d_ϕ is called the pseudo-metric induced by ϕ.

Proof. Obviously, $d_\phi(x,y) \geq 0$, $d_\phi(x,x) = 0$ and $d_\phi(x,y) = d_\phi(y,x)$ for any $x,y \in$ H. Using Proposition 3.3, we find that $d_\phi(x,y) + d_\phi(y,z) = (\phi(x \to y) + \phi(y \to x)) + (\phi(y \to z) + \phi(z \to y)) = (\phi(x \to y) + \phi(y \to z)) + (\phi(z \to y) + \phi(y \to x)) \geq \phi(x \to z) + \phi(z \to x) = d_\phi(x,z)$. Hence (H, d_ϕ) is a pseudo-metric space.

Proposition 3.12 Let ϕ be a pseudo valuation on H and d_ϕ the pseudo-metric induced by ϕ. Then the following results are valid: for any $x,y,z \in$ H,
(1) $\max\{d_\phi(x \to z, y \to z), d_\phi(z \to x, z \to y)\} \leq d_\phi(x,y)$;
(2) $d_\phi(x \wedge z, y \wedge z) \leq d_\phi(x,y)$;
(3) $d_\phi(x \otimes z, y \otimes z) \leq d_\phi(x,y)$.

Proof. (1) For any $x,y,z \in$ H, $y \to x \leq (x \to z) \to (y \to z)$ and $x \to y \leq (y \to z) \to (x \to z)$, then we get that $\phi(y \to x) \geq \phi((x \to z) \to (y \to z))$ and $\phi(x \to y) \geq \phi((y \to z) \to (x \to z))$. And so $d_\phi(x,y) = \phi(y \to x) + \phi(x \to y) \geq \phi((x \to z) \to (y \to z)) + \phi((y \to z) \to (x \to z)) = d_\phi(x \to z, y \to z)$. Analogously, $d_\phi(x,y) \geq d_\phi(z \to x, z \to y)$. Hence $\max\{d_\phi(x \to z, y \to z), d_\phi(z \to x, z \to y)\} \leq d_\phi(x,y)$.
(2)   Since $d_\phi(x \wedge z, y \wedge z) = \phi((x \wedge z) \to (y \wedge z)) + \phi((y \wedge z) \to (x \wedge z))$ for any $x,y,z \in$ H, and consider that $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$, we have that $\phi(x \to y) \geq \phi((x \wedge z) \to (y \wedge z))$ and $\phi(y \to x) \geq \phi((y \wedge z) \to (x \wedge z))$. And thus $d_\phi(x,y) = \phi(x \to y) + \phi(y \to x) \geq \phi((x \wedge z) \to (y \wedge z)) + \phi((y \wedge z) \to (x \wedge z)) = d_\phi(x \wedge z, y \wedge z)$.
(3)   In view of Proposition 2.2 (7), it suffices to show that $\phi((x \otimes z) \to (y \otimes z)) \leq \phi(x \to y)$ and $\phi((y \otimes z) \to (x \otimes z)) \leq \phi(y \to x)$ for any $x,y,z \in$ H. Hence $d_\phi(x,y) = \phi(x \to y) + \phi(y \to x) \geq \phi((x \otimes z) \to (y \otimes z)) + \phi((y \otimes z) \to (x \otimes z)) = d_\phi(x \otimes z, y \otimes z)$.

Proposition 3.13 Let ϕ be a pseudo valuation on H and d_ϕ the pseudo-metric induced by ϕ. Then

$(H \times H, d_\varphi^*)$ Is a pseudo-metric space, where

$$d_\varphi^*((x,y),(w,v)) = \max\{d_\varphi(x,w), d_\varphi(y,v)\},$$

For any (x,y),(w,v) ∈ H × H.

Proof. To complete the proof, we argue as follows: for any (x,y),(s,t),(w,v) ∈ H × H,

(1) $d_\phi^*((x,y),(x,y)) = \max\{d_\phi(x,x), d_\phi(y,y)\} = 0$.

(2) $d_\phi^*((x,y),(w,v)) = \max\{d_\phi(x,w), d_\phi(y,v)\} = \max\{d_\phi(w,x), d_\phi(v,y)\} = d_\phi^*((\omega,v),(x,y))$.

(3) $d_\phi^*((x,y),(s,t)) + d_\phi^*((s,t),(\omega,v)) = \max\{d_\phi(x,s), d_\phi(y,t)\} + \max\{d_\phi(s,w), d_\phi(t,v)\} \geq \max\{d_\phi(x,s) + d_\phi(s,w), d_\phi(y,t) + d_\phi(t,v)\} \geq \max\{d_\phi(x,\omega), d_\phi(y,v)\} = d_\phi^*((x,y),(w,v))$.

Consequently, $(H \times H, d_\varphi^*)$ is a pseudo-metric space.

Theorem 3.14 Let ϕ be a pseudo valuation on H and d_ϕ the pseudo-metric induced by ϕ. Then the operations ∧,⊗,→: H × H → H are uniformly continuous.

Proof. Here we only prove that ∧: H ×H → L is uniformly continuous, others can be proved similarly. For any x,y,w,v ∈ H and $\varepsilon > 0$, if $d_\varphi^*((x,y),(w,v)) < \frac{\varepsilon}{2}$, then $d_\varphi(x,w) < \frac{\varepsilon}{2}$ and $d_\varphi(y,v) < \frac{\varepsilon}{2}$. According to Proposition 3.12, we get $d_\phi(x \wedge y, w \wedge v) \leq d_\phi(x \wedge y, w \wedge y) + d_\phi(w \wedge y, w \wedge v) \leq d_\phi(x,w) + d_\phi(y,v) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence the operation ∧ is uniformly continuous.

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