EPH - International Journal of Mathematics and Statistics
ISSN (Online): 2208-2212
Volume 4 Issue 1 January 2018

DOI:https://doi.org/10.53555/eijms.v4i1.14

THE EXPONENTIAL MODIFIED WEIBULL LOGISTIC DISTRIBUTION (EMWL)

M. M. Nassar[1*], S. S. Radwan[2] and A. Elmasry[3]
*[1,3] *Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, 11566, Egypt*
[2]*Department of Mathematics, Faculty of Science, Al-Azhar University, (Girls' Branch), Nasr City, Cairo, Egypt*

Corresponding Author:-

Abstract:-
This paper introduces a new distribution named Exponential Modified Weibull logistic distribution. This distribution generalizes the following distributions: (1) Linear Failure Rate Logistic Distribution, (2) Weibull Logistic Distribution, (3) Rayleigh Logistic Distribution, (4) Exponential Logistic Distribution, where the failure rate, Weibull, Rayleigh and exponential distributions are the distributions most used for analyzing lifetime data. The properties of the new distribution are derived that include expressions for the r^{th}*moment, characteristic function and quantile function. The estimation of model parameters are performed by the method of maximum likelihood and hence evaluation of the performance of maximum likelihood estimation using simulation.*

Keywords:-*Modified Weibull distribution, Quantile function, and maximum likelihood estimation.*

1. INTRODUCTION

In this paper we introduce a generalization of the Logistic distribution via the modified Weibull distribution and the exponential generator.This leads to the Exponential Modified Weibull Lodistic distribution (EMWL) using the exponential generator applied to the odd ratio $\frac{G(x)}{\overline{G(x)}}$, where $[\overline{G(x)} = 1 - G(x)]$ such as the exponential Pareto distribution by AL-Kadim and Boshi (2013), exponential Lomax distribution by Bassiouny et al. (2015). The Weibull distribution is an important and popular distribution for modeling lifetime data. Recently, new classes of distributions were based on modifications of the Weibull distribution to provide a good fit to data sets. The two-parameter flexible Weibull extension was discusse d by Bebbington et al. (2007). Zhang and Xie (2011) studied the characteristics and application of the truncated Weibull distribution. Xie et al. (2002) proposed a three-parameter modified Weibull extension. The modified Weibull distribution was introduced by Lai et al. (2003). Recent studies of the modified Weibull include Jiang et al. (2010), Soliman et al. (2012) and Upadhyaya and Gupta (2010). Among the four-parameter distributions, the additive Weibull distribution of Xie and Lai (1996) with cdf

$$(X: \theta, a, b) = 1 - e^{-\theta x\eta - axb} \quad x \geq 0,$$

The Modified Weibull distribution that we use in our article of Sarhan and Zaindin (2009) can be derived from the additive Weibull distribution by setting $\eta = 1$. The article is outlined as follows. In Section 2 we introduce a new four-parameter distribution function called as Exponential Modified Weibull Logistic distribution with parameters a, b, λ and θ and it will be denoted as EMWL (a, b, λ, θ) and provide plots of density function (pdf) and cumulative distribution function (cdf), along with the hazard and survival function. It is observed that the EMWL (a, b, λ, θ) is a unimodal pdf and it has increasing hazard functions. Section 3 introduces some properties of the EMWL distribution as well as a complete discussion in deducing an implicit form for the Quantile function, numerical values for mode and explicit form for the characteristic function and moments followed by the deduction of Renyi and Shannon entropies. In Section 4 we derive the maximum likelihood estimators of the unknown parameters of the EMWL distribution. We present, in Section 5, a simulation study, followed, in Section 6, by an application to real data to illustrate the importance of the EMWL distribution.

2. The Exponential Modified Weibull Logistic Distribution (EMWL)

The cdf of the MWD (θ, a, b) derived by Sarhan and Zaindin(2009) takes the following form

$$F(X: \theta, a, b) = 1 - e^{-\theta x - ax^b} \quad x > 0,$$

where $b > 0$, $\theta, a \geq 0$, such that $\theta + a > 0$.

Here θ is a scale parameter, while a and b are shape parameters.

Then the exponential modified Weibull family obtain by replacing x with $\frac{1}{1-G(x)}$, we define the cdf family by

$$F(X: \theta, a, b) = 1 - e^{-\theta\left(\frac{1}{1-G(x)}\right) - a\left(\frac{1}{1-G(x)}\right)^b} \quad x > 0, \quad b > 0, \quad \theta, a \geq 0,$$

Taking $G(x)$ as the logistic distribution defined by

$$G(x) = \frac{1}{1 + e^{-\lambda x}}, -\infty < x < \infty, \lambda > 0.$$

Then the cdf of the EMWL $(X: \theta, a, b, \lambda)$ given by

$$F_{EMWL(}X: \theta, a, b, \lambda) = 1 - e^{-\theta(1+e^{\lambda x}) - a(1+e^{\lambda x})^b}, \ -\infty < x < \infty \ \ b, \lambda > 0, \ \theta, a \geq 0,$$
(1)

and the pdf of the EMWL $(X: \theta, a, b, \lambda)$ defined by

$$f_{EMWL(}(X: \theta, a, b, \lambda) = \lambda e^{\lambda x}\left[\theta + ab\left(1 + e^{\lambda x}\right)^{b-1}\right] e^{-\theta\left(1+e^{\lambda x}\right) - a\left(1+e^{\lambda x}\right)^b},$$

$$-\infty < x < \infty \quad b, \lambda > 0, \ \theta, a \geq 0, \qquad (2)$$

Plots of the pdf and cdf of the EMWL for different values of the parameters are given in Figures (1) and (2).

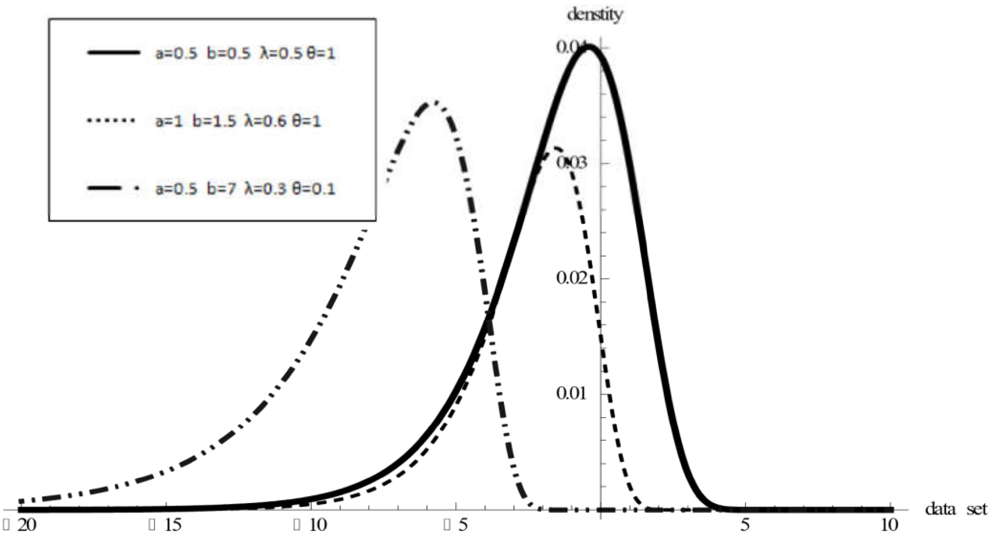

Figure (1). The pdf of the EMWL distribution for different values of the parameters.

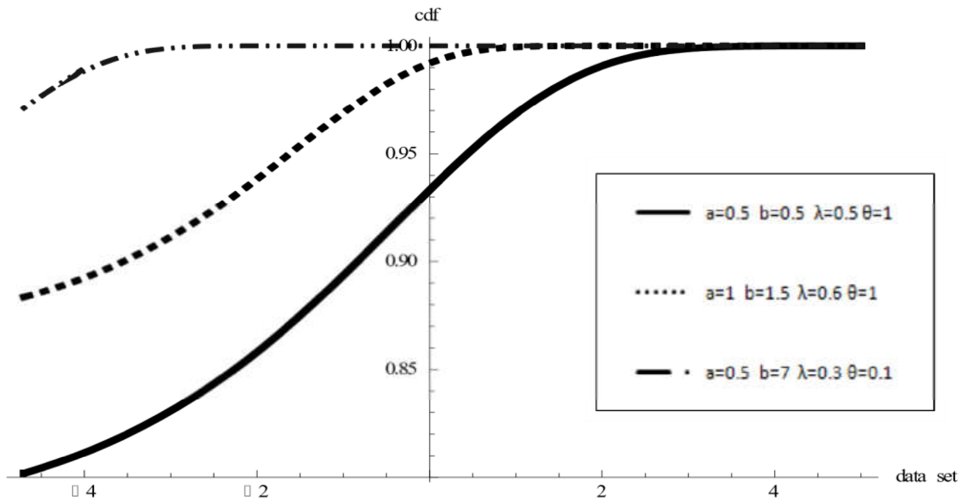

Figure (2). The cdf of the EMWL distribution for different values of the parameters.

$$h(x) = \frac{f(x)}{S(x)} =$$

$$= \lambda e^{\lambda x}\left[\Theta + ab\left(1 + e^{\lambda x}\right)^{b-1}\right] \qquad (3)$$

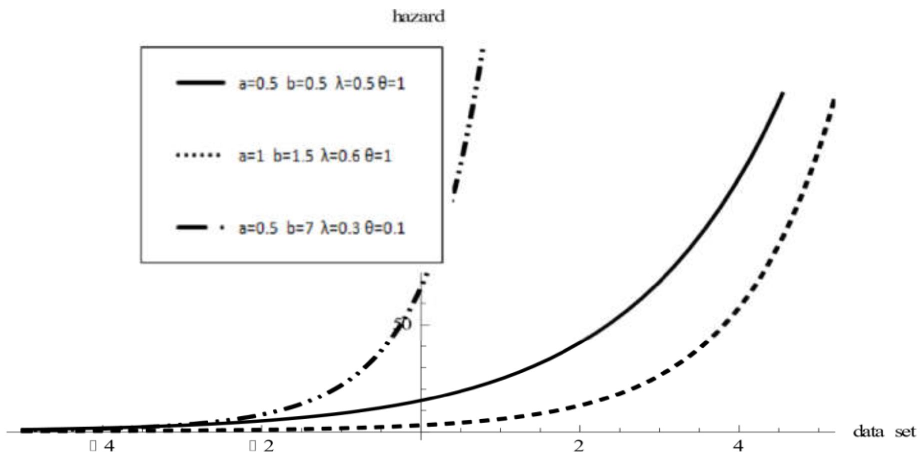

Figure (3): The hazard rate of the EMWL distribution for selected values of the parameters.

Let X be a random variable with density function (2), we write $X \sim EM\ (x;\ \theta,\ a,\ b,\ \lambda)$ Figure (1) shows the diverse shape of the EMWL pdf with different choice of parameters that include some well-known distributions. As the random variable $X \to \infty\ or - \infty$ the density of the EMWL distribution tends to zero. Figure (3) illustrates some possible shapes of the instantaneous failure rate function for some selected choices of parameters for the EMWL model. A characteristic of the EMWL distribution shows that the distribution has increasing hazard rate function for all choice of parameters. The EMWL distribution contains several well-known distributions as special cases when its parameters change. Table 1 demonstrates the sub-models of the EMWL distribution.

Table (1): Sub-models of the Exponential Modified Weibull Logistic distribution

Distribution	Θ	a	B	Λ
The Linear Failure Rate Logistic Distribution (LFRL)	-	-	2	-
The Weibull Logistic Distribution (WL)	0	-	- 2	-
The Rayleigh Logistic Distribution (RL)	0	-	- 1	-
The Exponential Logistic Distribution (EL)	-	0		-
The Exponential Logistic Distribution (EL)	-	-		-

Section 3 is devoted for studying statistical properties of $EM(\theta, a, b, \lambda)$ such as a complete discussion in deducing an implicit form for the Quantile function, numerical values for mode at different values of parameters and explicit expression for the characteristic function and r^{th} moment followed by the deduction of Renyi and Shannon entropies

3. Properties of the EMWL distribution

3.1. u^{th} Quantile

Theorem 1:

Let X be a random variable following $MW(\theta, a, b, \lambda)$ distribution and let uÎ (0,1). A value of x such that $F(x)=u$ is called a quantile of order u for the distribution. A quantile of order u is the real solution of the following equation

$$\text{Log}\,[1-u] + \theta(1+e^{\lambda x}) + a(1+e^{\lambda x})^{b} = 0$$

Proof:

Since (x) is continuous and strictly increasing, then the quantile function

$x = F^{-1}(u)$, $u \in (0, 1)$ can be straightforward computed by inverting (1) to obtain

$u=1-e-(1+e^{\lambda x})-a(1+e^{\lambda x})^{b}$ $1-u=e-\theta(1+e^{\lambda x})-a(1+e^{\lambda x})^{b}$

$\text{Log}\,(1-u) = -\theta\,(1+e^{\lambda x}) - a\,(1+e^{\lambda x})^{b}$

Therefore, an approximate Quantile function of order u of the EMWL distribution is the real solution of the equation in (4).

Equation (4) has no closed form solution in x, so we have to use a numerical tequnique .

Can use (4), to derive the following special cases:

(i) The u-th quantile of the LFRL(θ, a) distribution, by setting b=2, as

$$x = \frac{1}{\lambda}\ \log[(\frac{1}{2a} -\theta\ 1] + \sqrt{\theta^2 - 4a\log(1-u)}\Big) -$$

(ii) The u-th quantile of the WL(a, b) distribution, by setting θ =0, as

$$x = {}^{1\,-1\,1/}b - 1]\ \log[(\log(1-u))\ \frac{}{\lambda}\ \frac{}{a}$$

(iii) The u-th quantile of the RL(a) distribution, by setting b=2 and θ =0, as

$$x = \frac{1}{\lambda}\log[\sqrt{(}\frac{-1}{a}\log(1-u)) - 1]$$

(iv) The u-th quantile of the EL(θ) distribution, by setting a =0, as

$$x = \frac{1}{\lambda}\log\left[(\frac{-1}{\theta}\log(1-u)) - 1\right]$$

Or by putting b=1

$$x = \frac{1}{\lambda}\log\left[(\frac{-1}{(\theta+a)}\log(1-u)) - 1\right]$$

Put u=0.5 in equation (4) we get the median of EMWL (θ, a, b, λ).

The random sample can also be easily generated from (4) by taking U as a uniform random variable in (0, 1).

3.2. Mode

Mode is one of the most important characteristic features for the distribution. The mode of the $EMWL\ (\theta, a, b, \lambda,)$ is deduced by differentiating the pdf (1).

$$\frac{df(x; \theta, a, b, \eta, \lambda, t)}{dx}$$

$= \lambda^2 A e^{-\theta B - a}\,[ab(b-1)AB^{b-2} + [\theta + ab\theta B^{b-1}][1 - A(\theta + ab\theta B^{b-1})]]$
Where
A=e^{λ} & $B = 1 + A$.
Since A, B >0, then
$[(b-1)\,AB^{b-2} + [\theta + ab\theta B^{b-1}]\,[1 - A(\theta + ab\theta B^{b-1})]] = 0$
But we cannot obtain an explicit form so we calculate the mode numerically for different values of parameters

Table (2): Mode for some chosen different values of parameters.

Parameters	**Mode**
$a = 0.03 \quad b = 0.02 \quad \lambda = 0.5 \quad \theta = 0.1$	4.59943130379
$a = 0.5 \quad b = 0.5 \quad \lambda = 0.1 \quad \theta = 1$	−0.4120557810
$a = 0.5 \quad b = 0.4 \quad \lambda = 0.3 \quad \theta = 0.3$	2.704684469945
$a = 2.5 \quad b = 0.4 \quad \lambda = 0.03 \quad \theta = 0.03$	1.2206998906607
$a = 1 \quad b = 1.5 \quad \lambda = 0.6 \quad \theta = 1$	−1.5545128973
$a = 0.5 \quad b = 7 \quad \lambda = 0.3 \quad \theta = 1$	−5.7842349057

3.3. Characteristic Function
In this subsection, we derive the characteristic function (cf) of EMWL distribution.
Theorem 2:

$$\Phi_x(\rho) = \sum_{k,\ell=0}^{\infty} \frac{(-1)^{\ell} a^{\ell} e^{-\theta}}{\ell!\, \theta^{\frac{i\rho}{\lambda}+k+1}}\, \Gamma\left(\frac{i\rho}{\lambda} + k + 1\right)\left[\theta\binom{b\ell}{k} + ab\binom{b\ell + b - 1}{k}\right]$$
$$b > 0, \quad \theta, a \geq 0 \;\;, \lambda > 0 \;and\; -\infty < x < \infty. \qquad (5)$$

Proof:
We have the (cf) of the EMWL distribution as follows $\Phi_x(\rho) = E(e^{i\rho x}) =$

$$\int_{-\infty}^{\infty} e^{i\rho x}\;\; \lambda e^{\lambda x}\left[\theta + ab\left(1 + e^{\lambda x}\right)^{b-1}\right] e^{-\theta\left(1+e^{\lambda x}\right) - a\left(1+e^{\lambda x}\right)^{b}}\; dx$$

Setting $z = e^{\lambda x}$ and using the following expansions $e^{-a(1+z)^b}$ given by

$$e^{-a(1+z)^b} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(a)^{\ell}(1+z)^{b\ell}}{\ell!}, \qquad (*)$$

then

$$\Phi_x(\rho) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(a)^{\ell}}{\ell!}[\theta \int_0^{\infty} z^{\frac{i\rho}{\lambda}}(1+z)^{b\ell} e^{-\theta(1+z)}\, dz + ab \int_0^{\infty} z^{\frac{i\rho}{\lambda}}(1+z)^{b\ell+b-1} e^{-\theta(1+z)}]$$

Using the series expansion, the above equation reduces to

$$\Phi_x(\rho) = \sum_{k,\ell=0}^{\infty} \frac{(-1)^{\ell}(a)^{\ell}}{\ell!}[\theta\binom{b\ell}{k}\int_0^{\infty} z^{\frac{i\rho}{\lambda}+k} e^{-\theta(1+z)}\, dz$$
$$+ ab\binom{b\ell + b - 1}{k}\int_0^{\infty} z^{\frac{i\rho}{\lambda}+k} e^{-\theta(1+z)} dz]$$

0

By integrating the above equation we obtain the cf of the EMWL(x; θ, a, b, λ) distribution given by (5). We study the following cases.

(i) When $\theta = 0$, $a > 0$, then

$$\Phi_x(\rho) = ab\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(a)^{\ell}}{\ell!}\frac{\Gamma\left(\frac{i\rho}{\lambda}+1\right)\Gamma\left(-\frac{i\rho}{\lambda}-b\ell-b\right)}{\Gamma(1-b\ell-b)},$$

$$Re\left(\frac{i\rho}{\lambda}+b\ell+b\right) < 0 \;\&\; Re\left(\frac{i\rho}{\lambda}\right) > -1$$

(ii) When > 0, $a = 0$, we have

$$\Phi_x(\rho) = \frac{e^{-\theta}}{\theta^{\frac{i\rho}{\lambda}}}\Gamma\left(\frac{i\rho}{\lambda}+1\right), \qquad Re\left(\frac{i\rho}{\lambda}\right) > -1$$

3.4 r^{th} *Moment*

Theorem 3:

If X follows the EMWL(x; a, b, λ), then the r^{th} moment of X, μ_r, is given as follows

$$\mu_r = \frac{1}{(\lambda)^r}\sum_{k,\ell=0}^{\infty}\frac{(-1)^{\ell}(a)^{\ell}e^{-\theta}}{\ell!}\frac{\partial^r}{\partial(k+1)^r}\left(\frac{\Gamma(k+1)}{\theta^{k+1}}\right)\left[\theta\binom{b\ell}{k} + ab\binom{b\ell+b-1}{k}\right] \tag{6}$$

Proof:

We have the r^{th} moment of the EMWL distribution as follows

$\mu_r = E(x^r)$

$$= \int_{-\infty}^{\infty} x^r\ \lambda e^{\lambda x}\left[\theta\ + ab\left(1+e^{\lambda x}\right)^{b-1}\right]e^{-\theta\left(1+e^{\lambda x}\right)-a\left(1+e^{\lambda x}\right)^{b}}\,dx$$

Again setting $z = e^{\lambda x}$ and using the expansion (*), we obtain

$$\mu_r = \frac{1}{(\lambda)^r}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(a)^{\ell}}{\ell!}\left[\theta\int_0^{\infty}(\log z)^r\ (1+z)^{b\ell}e^{-\theta(1+z)}\,dz + ab\int_0^{\infty}(\log z)^r\,(1+z)^{b\ell+b-1}e^{-\theta(1+z)}\right]$$

Using the series expansion, the above equation reduces to

$$\mu_r = \frac{1}{(\lambda)^r}\sum_{k,\ell=0}^{\infty}\frac{(-1)^{\ell}(a)^{\ell}}{\ell!}\left[\theta\binom{b\ell}{k}e^{-\theta}\int_0^{\infty} z^k(\log z)^r\,e^{-\theta z}\,dz + ab\binom{b\ell+b-1}{k}e^{-\theta}\int_0^{\infty} z^k\,(\log z)^r\,e^{-\theta z}dz\right]$$

By integrating the above equation we obtain the r^{th} moment of the EMWL(x; θ, a, b, λ) distribution given by (6).

Based on the results given in (6), the measures of skewness and kurtosis of the EMWL distribution can be obtained according to the following relations, respectively,

$$\alpha^* = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^{\,3}}{(\mu_2-\mu_1^{\,2})^{\frac{3}{2}}}, \tag{7}$$

$$\beta^* = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^{\,2}\mu_2 - 3\mu_1^{\,4}}{(\mu_2-\mu_1^{\,2})^2}. \tag{8}$$

3.5. *Reni and Shannon entropies*

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. Since the entropy of a random variable is a measure of variation of the uncertainty, the Renyi entropy can be deduced to yield

$$I_X(\xi) = \frac{1}{1-\xi} \log[\lambda^{\xi-1} \sum_{k,\ell=0}^{\infty} \frac{(-1)^\ell (a)^\ell}{\ell!} \frac{e^{-\xi\theta}}{(\theta\xi)^{\xi+k}} \Gamma(\xi+k) \left[\theta \binom{b\ell}{k} + ab \binom{b\ell+b-1}{k}\right]]$$

$$\xi \geq 0, \xi \neq 1 \quad .(9)$$

A special case, defined in Shannon's [1948] pioneering work on the mathematical theory of communication, given by Shannon entropy - a major tool in information theory and in almost every branch of science and engineering is

$$h_{sh}(f_{TWL}) = -\log\lambda$$

$$-\frac{1}{\lambda} \sum_{k,\ell=0}^{\infty} \frac{(-1)^\ell (a)^\ell e^{-\theta}}{\ell!} \left(\frac{1}{\theta^{k+1}}\right) \left[\theta \binom{b\ell}{k} + ab \binom{b\ell+b-1}{k}\right] [\grave{\Gamma}(k+1) + \Gamma(1+k) \log\frac{1}{\theta}]$$

$$- \sum_{k,\ell=0}^{\infty} \frac{(-1)^\ell (a)^\ell}{\ell!} \Bigg[\left(\frac{1}{\theta}\right)^{b\ell} \log\theta \left[\Gamma(b\ell+1) + ab \left(\frac{1}{\theta}\right)^{b} \Gamma(b\ell+b)\right]$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{ab}{\theta}\right)^m \left(\frac{1}{\theta}\right)^{b\ell+bm-m} \Big[\Gamma(b\ell+bm-m+1)$$

$$+ ab \left(\frac{1}{\theta}\right)^{b} \Gamma(b\ell+b+bm-m)\Big]\Bigg]$$

$$+ \sum_{k,\ell=0}^{\infty} \frac{(-1)^\ell (a)^\ell}{\ell!} \Bigg[\left(\frac{1}{\theta}\right)^{b\ell} \Gamma(b\ell+2) + ab \left(\frac{1}{\theta}\right)^{b\ell+b} \Gamma(b\ell+b+1) + a \left(\frac{1}{\theta}\right)^{b\ell+b} \Gamma(b\ell+b+1)$$

$$+ a^2 b \left(\frac{1}{\theta}\right)^{b\ell+2b} \Gamma(b\ell+2b)\Bigg]$$

In Section 4 we discuss maximum likelihood estimation and in Section 5 we present a simulation study.

4. Maximum likelihood estimation

Here, we consider the maximum likelihood estimators (MLE) of the EMWL (θ, a, b, λ) distribution given in (2). Let X□(X_1, X_2, X_n) be a random sample of size n from this distribution. The log-likelihood function can be written as follows

$$\log L = n \log\lambda + \lambda \sum_{i=1}^{n} x_i - \theta \sum_{i=1}^{n} (1 + e^{\lambda x_i}) - a \sum_{i=1}^{n} (1 + e^{\lambda x_i})^b$$

$$+ \sum_{i=1}^{n} \log(\theta + ab(1 + e^{\lambda x_i})^{b-1}$$

Respectively, by taking the partial derivatives of the log- likelihood function with respect a, b, λ and θ, then equating it To zero, we obtain the estimating equations

$$\frac{\partial \mathrm{Log}L}{\partial a} = -\sum_{i=1}^{n} \left(1+e^{\lambda x_i}\right)^{b} + \sum_{i=1}^{n} \frac{b\left(1+e^{\lambda x_i}\right)^{-1+b}}{ab(1+e^{\lambda x_i})^{-1+b}+\theta},$$

$$\frac{\partial \mathrm{Log}L}{\partial b} = -a\sum_{i=1}^{n}(1+e^{\lambda x_i})^{b}\mathrm{Log}[1+e^{\lambda x_i}] + \sum_{i=1}^{n} \frac{a(1+e^{\lambda x_i})^{-1+b}+ab(1+e^{\lambda x_i})^{-1+b}\mathrm{Log}[1+e^{\lambda x_i}]}{ab(1+e^{\lambda x_i})^{-1+b}+\theta},$$

$$\frac{\partial \mathrm{Log}L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n} e^{\lambda x_i}x_i - a\sum_{i=1}^{n} be^{\lambda x_i}(1+e^{\lambda x_i})^{-1+b}x_i + \sum_{i=1}^{n} \frac{a(-1+b)be^{\lambda x_i}(1+e^{\lambda x_i})^{-2+b}x_i}{ab(1+e^{\lambda x_i})^{-1+b}+\theta},$$

$$\frac{\partial \mathrm{Log}L}{\partial \theta} = -\sum_{i=1}^{n}(1+e^{\lambda x_i}) + \sum_{i=1}^{n} \frac{1}{ab(1+e^{\lambda x_i})^{-1+b}+\theta},$$

The MLE can be determined numerically from the solution of nonlinear system of equations, subsequently; these solutions will yield the MLE estimators $\hat{a}$, $\hat{b}$, $\hat{\lambda}$ and $\hat{\theta}$. We required the observed information matrix for the interval estimation and hypothesis testing. For the four parameters EMWL distribution all the second order derivatives exist. Thus we have the observed information matrix as follows

$$K^{-1} = -E\begin{pmatrix} \frac{\partial^2 \log L}{\partial a^2} & \frac{\partial^2 \log L}{\partial a \partial b} & \frac{\partial^2 \log L}{\partial a \partial \lambda} & \frac{\partial^2 \log L}{\partial a \partial \theta} \\ \frac{\partial^2 \log L}{\partial b \partial a} & \frac{\partial^2 \log L}{\partial b^2} & \frac{\partial^2 \log L}{\partial b \partial \lambda} & \frac{\partial^2 \log L}{\partial b \partial \theta} \\ \frac{\partial^2 \log L}{\partial \lambda \partial a} & \frac{\partial^2 \log L}{\partial \lambda \partial b} & \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial a} & \frac{\partial^2 \log L}{\partial \theta \partial b} & \frac{\partial^2 \log L}{\partial \theta \partial \lambda} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The Fisher information matrix $K^{-1} = (\varphi)$, $\varphi = (a, b, \lambda, \theta)$.

Under conditions that are fulfilled for the parameter φ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $[\sqrt{n}(\hat{a}_{ML}-a), \sqrt{n}(\hat{b}_{ML}-b), \sqrt{n}(\hat{\lambda}_{ML}-\lambda), \sqrt{n}(\hat{\theta}_{ML}-\theta)]^T$ is $N_4(0, K^{-1}(a,b,\lambda,\theta)^T)$ defined by Miller (1981) The asymptotic normal $N_4\left(0, K^{-1}(\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{\theta}_{ML})^T\right)$ distribution of $\hat{\varphi} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{\theta}_{ML})^T$ can be used to construct confidence regions for some parameters and for the hazard and survival functions. In fact, a $100(1-\gamma)\%$ asymptotic confidence interval (ACI) for each parameter is given by

$$ACI_a = \left(\hat{a}_{ML} - z_{\gamma/2}\sqrt{K_{11}}, \hat{a}_{ML} + z_{\gamma/2}\sqrt{K_{11}}\right),$$

$$ACI_b = \left(\hat{b}_{ML} - z_{\gamma/2}\sqrt{K_{22}}, \hat{b}_{ML} + z_{\gamma/2}\sqrt{K_{22}}\right),$$

$$ACI_\lambda = \left(\hat{\lambda}_{ML} - z_{\gamma/2}\sqrt{K_{33}}, \hat{\lambda}_{ML} + z_{\gamma/2}\sqrt{K_{33}}\right),$$

$$ACI_\theta = \left(\hat{\theta}_{ML} - z_{\gamma/2}\sqrt{K_{44}}, \hat{\theta}_{ML} + z_{\gamma/2}\sqrt{K_{44}}\right),$$

Where K_{ii} denotes the i^{th} diagonal element of $K^{-1} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{\theta}_{ML})^T$ for I = 1, 2, 3, 4 and $z_{\square/2}$ is the (1□□/2) of the standard normal distribution.

5. Simulation Study

We conducted Mont Carlo simulation studies to assess the finite sample behavior of the EMWL (a, b, λ, θ) all results were obtained from 1000 Mont Carlo replication simulations. The EMWL random number generation was performed using the inversion method. In each replication, random sample of size n is drawn from the EMWl (a, b, λ, θ) distribution and the maximum likelihood estimates (MLEs) of the parameters were obtained. The mean, variance, bias and mean squared error (MSE) for each parameter was computed under different sample size n=10, 25, 75, 100, and 200.

Table (3): Mean estimates, bias, variance and mean square errors of the (MLEs) when $a = 2$, b = 3, $\lambda = 0.03$, Θ = 0.012.

N	parameter	Mean	Variance	Bias	MSE
10	a b λ Θ	4.606577 5 1.865921 9 0.302560 2 0.104531 7	8.284913655 11.7264116 0.038825932 0.002317102	2.6065 7 - 1.1341 0.2725 6 0.0925 3	15.079 2 13.012 5 0.1131 1 0.0108 8
25	a b λ Θ	3.66414 2.3480497 5 0.1077023 8 0.0830312 5	1.275658876 6.317470551 0.018200814 0.005815521	1.6641 4 - 0.6519 -0.0777 0.0710 3	4.0450208 6.7425096 0.0242384 0.0108609
75	a b λ Θ	4.4440625 4.7534925 0.01502036 8 0.03985742 3	0.107298029 0.02655224 0.0000278341 0.000318574	2.4440625 1.7534925 -0.01497963 0.02973075	6.0807395330.532723 3.1012881883.351777 0.000252223-0. 079 0.001202491-1. 365
100	a b λ Θ	4.5057575 4.8892675 0.01700222 5 0.03949815	0.085169826 0.029718544 0.000018083 0.0000843884	2.5057575 1.8892675 - 0.01299777 0.02749815	6.363990475 3.59905023 0.000187025 0.000840537
200	a b λ Θ	3.5125975 2.63189 0.01318597 5 0.01945587	0.052888346 0.010677358 0.0000058813 5 0.0000259771	1.5125975 -0.36811 - 0.01681402 0.00745587	2.380839543 0.14618233 0.000288593 0.0000815671

The mean estimates of the parameters tend to be closer to the true parameter values. It is observed that for all values of n, the variance and MSE of the estimators of a, b, λ and Θ are small as expected. We conclude, in Section 6, and an application to real data.

6. Numerical example

In this section we provide a data analysis to see how the Exponential Modified Weibull logistic distribution works in practice. The data have been obtained from Aarset (1987) and it is provided below. It represents the lifetimes of 50 devices.

0.1,0.2,1,1,1,1,1,2,3,6,7,11,12,18,18,18,18,18,21,32,36,40,45,46,47,50,55,60,63,63,67,67,67,67,72,75,79,82,82,83,84,84,84,85,85,85,85,85,86,86 .

For the data, we fit the Exponential Modified Weibull Logistic distribution (EMWL) defined in (2) and compare it with Modified Weibull distribution (MW) (for $0 < x < \infty$), Transmuted New Generalized Inverse Weibull Logistic distributions (TNGIWL) (for $-\infty < x < \infty$), Transmuted Weibull Logistic distribution (TWL) (for $-\infty < x < \infty$) and Transmuted Kumaraswamy Logistic distribution (TKL) (for $-\infty < x < \infty$) models with corresponding densities:

$$f_{MW} = (\theta + abx^{b-1})e^{-\theta x - ax^{b}} \quad x > 0, \qquad b > 0 \ \& \ a, \theta \geq 0$$

$$f_{TNGIWL}(x) = \eta\lambda(\theta e^{-\lambda x} + abe^{-b\lambda x})\left(e^{\{-\theta e^{-\lambda x} - ae^{-b\lambda x}\}}\right)\left(1 - e^{\{-\theta e^{-\lambda x} - ae^{-b\lambda x}\}}\right)^{\eta-1}$$

$$\left[1 - t + 2t\left(1 - e^{\{-\theta e^{-\lambda x} - ae^{-b\lambda x}\}}\right)^{\eta}\right], \quad \theta, a, b, \eta > 0, \lambda > 0, |t| \leq 1 \ and - \infty < x < \infty.$$

$$f_{TWL}(x) = ab\lambda e^{(\lambda bx - ae^{\lambda bx})}\left[1 - t + 2te^{-ae^{\lambda bx}}\right],$$

$$a, b > 0, \lambda > o \ and \ |t| \leq 1, -\infty < x < \infty.$$

$$f_{TKL}(x) = \left[\frac{ab\lambda e^{-\lambda X}}{(1 + e^{-\lambda X})^{a+1}}\left[1 - \left(\frac{1}{1 + e^{-\lambda X}}\right)^{a}\right]^{b-1}\left[1 - t + 2t\left[1 - \left(\frac{1}{1 + e^{-\lambda X}}\right)^{a}\right]^{b}\right]\right]$$

$$a, b > 0, \lambda > o \ and \ |t| \leq 1, \quad -\infty < x < \infty$$

The maximum likelihood method is applied to estimate the parameters of the five models Exponential Modified Weibull Logistic distribution (EMWL), Modified Weibull distribution (MW), Transmuted New Generalized Inverse Weibull Logistic distributions (TNGIWL), Transmuted Weibull Logistic distribution (TWL) and Transmuted Kumaraswamy Logistic distribution (TKL). The resulting estimates with the negative of the likelihood function ($-\ell$).

Table (4). The maximum likelihood estimated and Log- likelihood function for the data set.

Model	maximum likelihood estimates	$-\ell$
MW	$\hat{a} = 2.159 \times 10^{-8}$ $\hat{b} = 4.014$ $\hat{\Theta} = 0.012$	127.497
EMWL	$\hat{a} = 0.0031$ $\hat{b} = 2.3636$ $\hat{\lambda} = 0.0352$ $\hat{\Theta} = 0.1200$	108.988
TNGIWL	$\hat{a} = 3.41556$ $\hat{b} = 1.43833$ $\hat{\lambda} = 0.01204$ $\hat{t} = 0.98252$ $\hat{\Theta} = 0.52687$ $\hat{\eta} = 1.28252$	327.239
TWL	$\hat{a} = 0.26588$ $\hat{b} = 2.26213$ $\hat{\lambda} = 0.05786$	244015
	$\hat{t} = 0.03000$	
TKL	$\hat{a} = 2.42574$ $\hat{b} = 0.195187$ $\hat{\lambda} = 0.089133$ $\hat{t} = 0.568360$	374.5197

Table (5): Criteria comparison for the data set

Model	$-\ell$	AIC	AICC	BIC
MW	127.497	260.994	261.51574	260.0909
TNGIWL	327.239	666.478	668.43149	664.6718
TWL	244015	488038	488038.88889	488036.7959
TKL	374.5197	757.0394	757.92829	755.8353
MWL	108.988	225.976	226.86489	224.77188

The variance covariance matrix $(\varphi)^{-1}$ of the MLEs under the EMWL distribution for the data set is computed as

$$\begin{pmatrix} 5.37508 \times 10^{-7} & -2.63722 \times 10^{-5} & -5.23231 \times 10^{-7} & 2.53754 \times 10^{-6} \\ -2.63722 \times 10^{-5} & 0.015875700000 & -2.16827 \times 10^{-4} & -3.56274 \times 10^{-5} \\ -5.23231 \times 10^{-77} & -2.16827 \times 10^{-4} & 4.11029 \times 10^{-6} & -5.05316 \times 10^{-6} \\ 2.53754 \times 10^{-6} & -3.56274 \times 10^{-5} & -5.05316 \times 10^{-6} & 6.80931 \times 10^{-5} \end{pmatrix}$$

Thus $var\ (\hat{a}) = 5.37508 \times 10^{-7}, var\ (\hat{b}) = 0.0158757, var(\hat{\lambda}) = 4.11029 \times 10^{-6},$ $var\ (\hat{\theta}) = 6.80931 \times 10^{-5}.$

There for, 95% confidence interval for a, b, λ $and\ \theta$ are [0.001615, 0.004489], [2.116642, 2.610558], [0.031246, 0.039194], [0.103826, 0.136173] respectively.

In order to compare the five distributions, we consider criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (the Bayesian information criterion) for the data. As shown in table (5), the better distribution corresponds to smaller $-\ell$, AIC, AICC and BIC values

$$AIC = 2K - 2\ell$$

$$\mathrm{AICC} = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k \log n - 2\ell,$$

Here k is the number of parameters and n is the number of observations. The values of the parameters' estimates are used to plot the pdf for the five distributions EMWL, MW, TNGIWL, TWL and TKL in Fig (4)

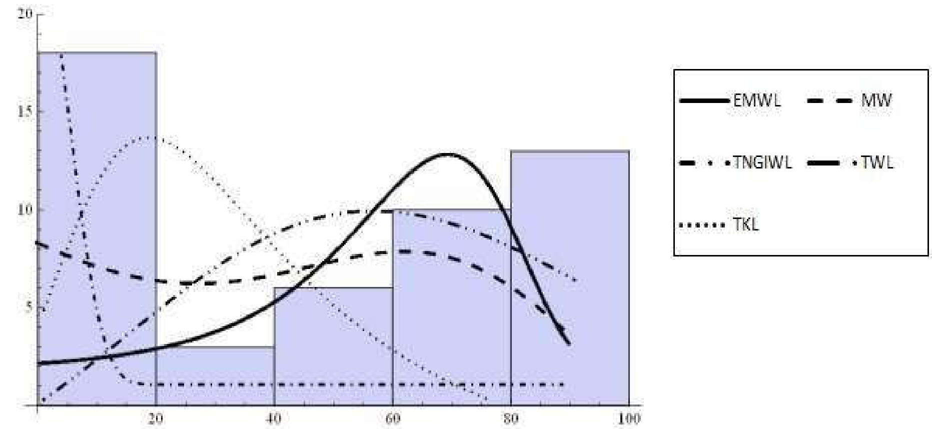

Figure (4). Estimated densities of the models for the data set.

7. Concluding Remarks

In this paper, we proposed a new distribution, named the Exponential Modified Weibull logistic distribution which extends the Modified Weibull distribution. Several properties of the new distribution were researched, including mode, Rényi and Shannon entropy and implicit expression for the quantile function, characteristic function and r^{th} moment. The new extended model has an increasing hazard rate function. The model parameters are estimated by maximum likelihood. An application of the Exponential Modified Weibull logistic distribution (EMWL) to real data is considered. The results of our study indicate that the EMWL distribution has the lowest AIC, AICC and BIC statistics among all the sub-models. From the plots of the fitted densities and histogram, clearly, the EMWL distribution provides a closer fit to the histogram than the other Modified Weibull (MW), Transmuted New Generalized Inverse Weibull logistic (TNGIWL), Transmuted Weibull Logistic distribution and Transmuted Kumaraswamy Logistic model. Therefore, the new EMWL model can be used quite effectively in analyzing data. Also, we note that the Monte Carlo simulation indicates that the performance of the maximum likelihood estimation is quite satisfactory. Finally, the application to the real data sets shows that the fit of the new model is superior to the fits of its main sub- models. We hope that the proposed model can be used effectively as a competitive model to fit real data.

References

[1]. Aarset, M.V. (1987), "How to identify bathtub hazard rate", IEEE Transactions on Reliability, 36(1), 106-108.

[2]. AlKadim, K.A. and Boshi, M.A. (2013), "Exponential Pareto Distribution", Mathematical Theory and Modeling, **3**, 135-146.

[3]. Bassiouny, A.H. Abdo, N.F. and Shahen, H.S. (2015), "Exponential Lomax Distribution", International Journal of Computer Applications, **13,** 24-29.

[4]. Bebbington, M., Lai, C. D. and Zitikis, R. (2007)," A flexible Weibull extension", Reliability Engineering and System Safety, 92, 719-726.

[5]. Jiang, H., Xie, M. and Tang, L. C. (2010), "On MLEs of the parameters of a modified Weibull distribution for progressively type-2 censored samples", Journal of Applied Statistics, 37(4), 617627.

[6]. Lai, C. D., Xie, M. and Murthy, D. N. P. (2003), "A modified Weibull distribution", IEEE Transactions on Reliability, 52(1), 33-37.

[7]. Miller, Jr., R.G. (1981), Survival Analysis, John Wiley, New York.

[8]. Sarhan, A.M. and Zaindin, M. (2009), "Modified Weibull distribution", Applied Sciences, **11,** 123-136.

[9]. Shannon,C.E. (1948), "A mathematical theory of communication", Bell Syst. Tech. J **27**, 379- 432.

[10]. Soliman, A. A., Abd-Ellah, A. H., Abou-Elheggag, N. A. and Ahmed, E. A. (2012), " Modified Weibull model: A Bayes study using MCMC approach based on progressive censoring data", Reliability Engineering and System Safety.

[11]. Upadhyay, S. K. and Gupta, A. (2010), " A Bayes analysis of modified Weibull distribution via Markov chain Monte Carlo simulation", Journal of Statistical Computation and Simulation, 80(3), 241-254.

[12]. Xie, M. and Lai, C. D. (1996), " Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function", Reliability Engineering and System Safety, 52(1), 8793.

[13]. Xie, M., Tang, Y. and Goh, T. N. (2002), "A modified Weibull extension with bathtub shaped failure rate function", Reliability Engineering and System Safety, 76(3), 279-285.

[14]. Zhang, T. and Xie, M. (2011), "On the upper truncated Weibull distribution and its reliability implications", Reliability Engineering and System Safety, 96(1), 194-200.