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COMPOUND RAYLEIGH LIFETIME DISTRIBUTION-I

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Abstract:-
This paper provides a new lifetime distribution model named as Compound Rayleigh Lifetime Distribution -I which is derived from Rayleigh distribution by imposing the restriction on the distribution of its parameter towards obtaining the properties of fine tuning manoeuvres and shift process in mean which is useful in reliability and statistical quality control analysis. The theoretical results obtained are illustrated with appropriate practical example.

Keywords:-*Rayleigh distribution, Reliability function, Shift in process mean, fine tuning manoeuvres.*

INTRODUCTION

The thrust of the present paper is to emphasize and bridge a cause and effect relationship between these two aspects of industrial practice. Based on process capability analysis, and the effects of certain trial and error fine-tuning manoeuvres on the part of the producer, leading to certain shifts in process mean (target value), the lifetime distribution of the manufactured product will no more adhere to the conventional models like exponential, gamma, Weibull, etc., discussed in the literature. The results obtained in this paper pertain to the derivation of a new lifetime model taking into consideration such modifications and manoeuvres. The properties of this model so derived is useful in reliability and SQC analysis.

The theoretical results obtained are illustrated with appropriate practical example.

1. COMPOUND DISTRIBUTIONS

In applications of statistics, one generally has some apriori information regarding variation in one or more of the parameters of the probability model under consideration such variation also is specified by means of a probability distribution on an appropriate support. When this information about the parameter(s) is made effective and is super imposed on the basic model, and the probability density function is derived, as a marginal, the resulting distribution is termed as the compound distribution of the basic model.

2. Compound Rayleigh lifetime distribution (CRLD) Model -I

Basic assumptions required for the derivation of the model are

1. The lifetime (T) of a product is Rayleigh with density f(t) given by

$$f(t) = \begin{cases} \dfrac{t}{\alpha^2} e^{-(1/2(t/\alpha)^2)} & ; \quad 0<t<\infty, \\ 0 & ; \quad elsewhere. \end{cases} \qquad (3.1)$$

The measurable quality characteristic X of the product is normally distributed with mean μ and variance σ^2 i.e., $X \sim N(\mu, \sigma^2)$, in which, σ^2 is known.

Let U, L represent the upper and lower specification limits prescribed by the designing department. Based on the observation of the production process over time, suppose it is concluded that U-L > 6 σ, implying that the process is capable of producing better products, meeting the specifications.

Under this framework, suppose the manufacturer decides to relax certain constraints on the process by fine-tuning the operations of either one or more of the three M's (material, men and machines).

2. The shift in μ resulting out of fine-tuning follows uniform distribution in [□L, □U], where

□U = U - 3□ and □L = L + 3□ (3.2)

It can be observed that the greater the shift in μ which results in a greater deviation from the process mean μ_o, will cause a reduction in the lifetime of the product. This, in turn, results in a reduction in the expected lifetime of the product.

3. The shift in μ resulting out of fine-tuning follows uniform distribution in $[\mu_L, \mu_U]$, where

$$\mu_U = U - 3\sigma \text{ and } \mu_L = L + 3\sigma \qquad (3.2)$$

It can be observed that the greater the shift in μ which results in a greater deviation from the process mean μ_o, will cause a reduction in the lifetime of the product. This, in turn, results in a reduction in the expected lifetime of the product.

4. The increase in the absolute deviation of μ from μ_o results in an increase in the reciprocal $\sqrt{2/\pi}\,\sigma^{-1}$ of the expected lifetime and, the same is represented by the relation

$$\sqrt{2/\pi}\ \sigma^{-1} = c + mU, \quad \text{where } c, m > 0 \qquad (3.3)$$

where $U = |\mu - \mu_o|$ which represents absolute deviation of μ from μ_o.

Lemma 1.1: The variation in the random variable U is specified by the probability density function (pdf) function (pdf)

$$g^*(u) = \begin{cases} \delta^{-1} & ; \quad 0 < u < \delta, \\ 0 & ; \quad \text{elsewhere.} \end{cases} \qquad (3.4)$$

where $\delta = 2^{-1}(\mu_U - \mu_L)$.

Proof: From the Assumption 3, the pdf of μ is

$$g(\mu) = \begin{cases} (\mu_U - \mu_L)^{-1} & ; \quad \mu_L < \mu < \mu_U, \\ 0 & ; \quad \text{elsewhere.} \end{cases} \qquad (3.5)$$

Let, μ_o be the target value. Then we have $\mu_o = (\mu_U + \mu_L)/2$.

Probability of μ lying between (μ_L, μ_o) is same as probability of μ lying between (μ_o, μ_U) and is equal to ½ . Then we can obtain

$$g^*(u)\, du = P(\mu_L < \mu < \mu_o)\frac{du}{\mu_o - \mu_L} + P(\mu_o < \mu < \mu_U)\frac{du}{\mu_U - \mu_o} = \frac{du}{\delta}.$$

For $0 < u < \delta$. Hence, the pdf of U is given by

$$g^*(u) = \begin{cases} \delta^{-1} & ; \quad 0 < u < \delta, \\ 0 & ; \quad \textit{elsewhere.} \end{cases}$$

Theorem 1.1: The distribution of the lifetime T under the framework as explained above, is given by the compound distribution with density $f^*(t)$ as

$$f^*(t) = \begin{cases} (m\delta)^{-1}\int_c^{c+m\delta} \pi t v^2 2^{-1} e^{-(t^2 v^2 \pi 4^{-1})} dv & ; \quad 0 < t < \infty, \\ 0 & ; \quad \textit{elsewhere.} \end{cases} \qquad (3.6)$$

Proof Using the Lemma 1.1, and affecting the transformation

$\sqrt{2/\pi}\, \sigma^{-1} = c + mU = V$ (say), then we have

$$f_1(v) = \begin{cases} (m\delta)^{-1} & ; \quad c < v < c + m\delta, \\ 0 & ; \quad \textit{elsewhere.} \end{cases} \qquad (3.7)$$

From the theory of compound distributions, one has the joint density function of V and T, h (v, t) of the compound distribution as

$$h(v, t) = f_1(v) \,.\, f(t/v)$$

$$= \begin{cases} (2m\delta)^{-1}\pi . t v^2 e^{-(\pi/4)t^2v^2} & ; \quad 0 < t < \infty, \; c < v < c + m\delta, \\ 0 & ; \quad \textit{elsewhere.} \end{cases} \qquad (3.8)$$

Thus, the lifetime density of T in (3.6) is obtained as the marginal density, byintegrating h(v, t) w.r.t. v, in the appropriate range.

Note: The pdf $f^*(t)$ can also be expressed in terms of incomplete gamma distribution function Γ, as follows

$f^*(t) = 2(m\delta)^{-1} t^{-2} \pi^{-1/2} [\Gamma(3/2,\omega_1) - \Gamma(3/2,\omega_2)]$; $0 < t <$ □ ,

0 ; elsewhere. (3.9) where

$\omega_1 = t^2 c^2 \pi 4^{-1}$ and $\omega_2 = t^2 (c + m\delta)^2 \pi 4^{-1}$ (3.10)

However, for convenience, the form (3.6) is adopted throughout the paper.

Lemma 1.2: The function $f^*(t)$ defined in (3.6) is a proper probability density function. **Proof**: Follows from (i) $f^*(t)$ should be non-negative, which is obvious, since the integrand in $f^*(t)$ is positive as c, m, δ are positive. (ii) Total probability is given by

$$I = \int_0^\infty f^*(t)\,dt = \int_0^\infty (m\delta)^{-1}\left[\int_c^{c+m\delta} \pi t v^2 2^{-1} e^{-t^2 v^2 \pi 4^{-1}}\,dv\right]dt = 1$$

PROPERTIES OF CRLD MODEL - I

Lemma 1.3: The expected value of T for CRLD model - I is given by

$$E^*(T) = (m\delta)^{-1}\log[1 + m\delta\, c^{-1}]\,. \tag{4.1}$$

Proof: From the definition of mathematical expectation, one has

$$E^*(T) = \int_0^\infty t\, f^*(t)\,dt = (m\delta)^{-1}\int_0^\infty t\left[\int_c^{c+m\delta}\frac{\pi t v^2}{2} e^{-1/2\frac{t^2 v^2 \pi}{2}}\,dv\right]dt$$

By changing the order of integration, and E*(T) on simplification yields (4.1).

Lemma 1.4 For all m, c>0, the expected lifetime of the CRLD model-I is less than the expected lifetime c^{-1} of the Rayleigh distribution.

Proof From (4.1),

$$E^*(T) = (m\delta)^{-1}\log(1 + m\delta c^{-1}) < \frac{1}{m\delta}\cdot\frac{m\delta}{c}$$

$$\text{since } \log(1+x) < x \text{ for } x \geq 0 \tag{4.2}$$

Hence, the assertion.

Lemma 1.5: The variance of CRLD model-I is

$$V^*(T) = [4(\pi c(c+m\delta))^{-1}] - [(m\delta)^{-2}(\log(1+m\delta c^{-1}))^2] \tag{4.3}$$

Proof: $V^*(T) = E^*(T^2) - (E^*(T))^2$, one has

$$E^*(T^2) = \int_0^\infty t^2 f^*(t)\,dt = \int_0^\infty t^2 (m\delta)^{-1}\left[\int_c^{c+m\delta}\pi t v^2 2^{-1} e^{-t^2 v^2 \pi 4^{-1}}\,dv\right]dt$$

By changing the order of integration

$$E^*(T^2) = (m\delta)^{-1}\int_c^{c+m\delta} 4(v^2\pi)^{-1}\,\Gamma(2)\,dv = 4(\pi c(c+m\delta))^{-1} \tag{4.4}$$

From (4.1) and (4.4), we can obtain

$$V^*(T) = [4(\pi c(c+m\delta))^{-1}] - [(m\delta)^{-2}(\log(1+m\delta c^{-1}))^2].$$

Lemma 1.6: The distribution function F*(t) of the CRLD model-I is given by

$$F^*(t) = 1-(m\delta t)^{-1}\pi^{-1/2}\int_{\omega_1}^{\omega_2} e^{-\omega}\,\omega^{1/2-1}\,d\omega \qquad (4.5)$$

where ω_1, ω_2 are as in (3.10).

Proof $\quad F^*(t) = \int_0^t f^*(x)\,dx = \int_0^t (m\delta)^{-1}\left[\int_c^{c+m\delta} \pi x v^2 2^{-1} e^{-x^2v^2\pi 4^{-1}}\,dv\right]dx$

By changing the order of integration and $x^2 v^2 \pi/4 = y$, $dy = 2xv^2\pi/4\,dx$, one has

$$F^*(T) = (m\delta)^{-1}\int_c^{c+m\delta} \pi v^2 2^{-1}\left[\int_0^{t^2v^2\pi 4^{-1}} e^{-y}\,2dy(v^2\pi)^{-1}\right]dv$$

$$= (m\delta)^{-1}\int_c^{c+m\delta}\left(1-e^{-t^2v^2\pi 4^{-1}}\right)dv$$

Let $\omega = t^2 v^2 \pi 4^{-1}$ so that $d\omega = 2t^2 v\pi 4^{-1}\,dv$

$$F^*(T) = (m\delta)^{-1}.m\delta-(m\delta)^{-1}\int_{\omega_1}^{\omega_2} e^{-\omega}2d\omega(t^2\pi)^{-1}.(t^2\pi)^{1/2}(4\omega)^{-1/2}$$

$$= 1-(m\delta t)^{-1}(\pi)^{-1/2}\int_{\omega_1}^{\omega_2} e^{-\omega}\,\omega^{1/2-1}\,d\omega$$

Note: An alternative form for the distribution function of CRLD-I can be expressed as

$$F^*(t) = 1-(m\delta t)^{-1}\pi^{-1/2}\,[\Gamma(1/2,\omega_1)-\Gamma(1/2,\omega_2)]$$

EXAMPLE 4.1: The case of manufacturing piston rings for an automotive engine by using certain forging process (Montgomery (1991), pp.206) is taken for the purpose of illustration of the theory developed in the preceding sections. The inside diameter of the ring manufactured by the process is the measurable quality characteristic X, assumed to be normally distributed with mean μ, variance σ^2. The upper and lower specification limits U and L, respectively, are U = 75 mm, L = 73 mm with the process, the target value μ_o = 74.001 mm and the estimated value of σ = 0.00989 mm. Thus, the process is observed to use up only about 60% of the specification band.
Using the theory of modified control charts the following values are obtained.
μ_U = 74.97033 mm, μ_L = 73.02967 mm and δ = 0.97033 mm

Table 4.1: Expected Lifetime of the CRLD model - I

m \ c	0.5	1	1.5	2	2.5	3	M=E(T) = 1/c
0.5	1.3975	1.11161	0.9369	0.8169	0.7252	0.6596	2
1	0.8152	0.6989	0.6171	0.555	0.5075	0.4684	1
1.5	0.5776	0.5141	0.4659	0.4277	0.3966	0.3705	0.6666
2	0.4476	0.4076	0.3756	0.3494	0.3274	0.3085	0.5
2.5	0.3655	0.3379	0.3152	0.2960	0.2795	0.2652	0.4
3	0.3089	0.2888	0.2717	0.2570	0.2442	0.2329	0.3333

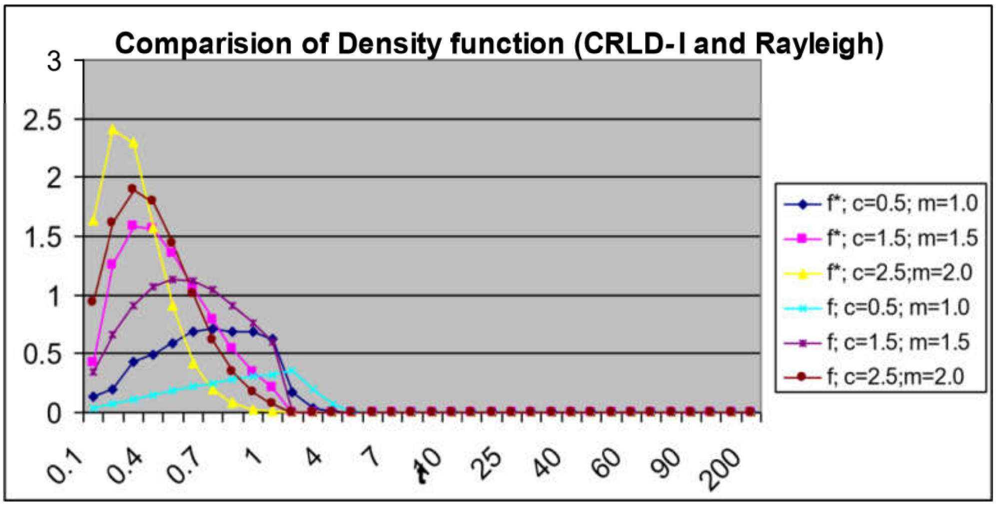
From the table 4.1, it can be observed that the values of expected lifetime under CRLD-I is smaller when compared to that (M) of the Rayleigh lifetime distribution. It can also be observed rate of this fall in the expected lifetime is decreasing as either each of c or m is increasing (along rows and along columns).
f^*, F^*, f, F represent the characteristics density function and distribution function for CRLD model - I and conventional Raleigh Lifetime Distribution respectively and are tabulated in Tables 4.2 for the example considered. Further, their respective graphs are also presented in Graphs 1, 2

Table 4. 2: CRLD model - I and Rayleigh Distribution

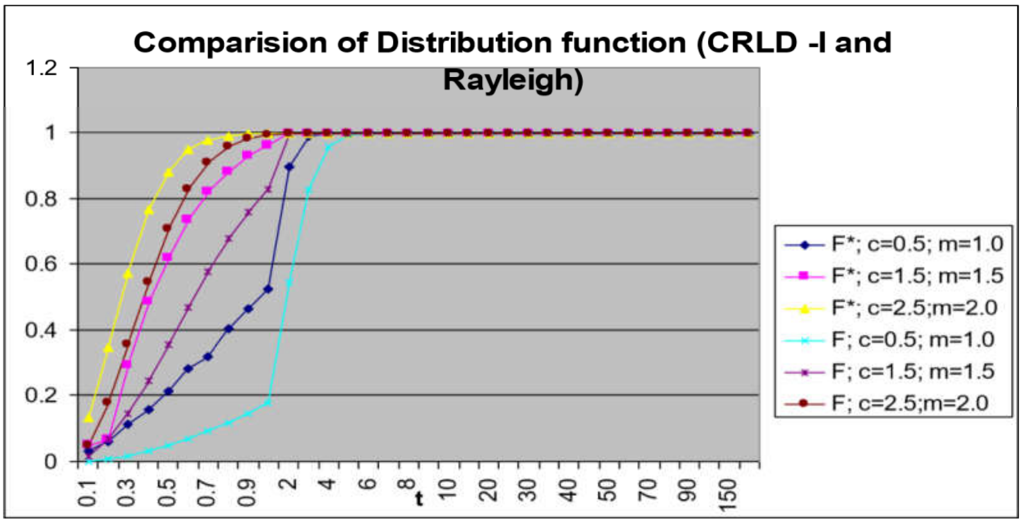
	C= 0.50000 and m= 1.00000				c= 1.50000 and m=1.500000			
t	f*	F*	f	F	f*	F*	f	F
0.1	0.13395	0.033	0.0392	0.002	0.416	0.05	0.347	0.0175
0.2	0.190884	0.06	0.0779	0.007	1.2533	0.064	0.6581	0.0682
0.3	0.433576	0.115	0.1157	0.0178	1.580	0.2895	0.9043	0.147
0.4	0.49762	0.1570	0.1521	0.0305	1.5654	0.483	1.065	0.2462
0.5	0.586889	0.2155	0.1869	0.0479	1.350	0.618	1.1351	0.357
0.6	0.683729	0.2812	0.2194	0.0689	1.0636	0.7375	1.1228	0.4705
0.7	0.71647	0.3195	0.2496	0.0912	0.782	0.82	1.0402	0.5791
0.8	0.68932	0.4058	0.2769	0.1187	0.5369	0.88	0.9127	0.6771
0.9	0.682633	0.4646	0.3013	0.147	0.3496	0.928	0.7605	0.7608
1	0.6248	0.524	0.3226	0.178	0.2092	0.96	0.6043	0.829
2	0.16894	0.896	0.3581	0.5432	0.0001	0.999	0.006	0.9991
3	0.03527	0.988	0.2013	0.8299	40	9 1	0	1
4	0.00658	0.9961	0.068	0.956	0	1	0	1
5	0.00083	0.9997	0.0145	0.9927	0	1	0	1
6	0.000088	0.9996	0.002	0.9996	0	1	0	1
7	5.6E-063	0.9999	0.0002	0.9991	0	1	0	1
8	3.22E-	91	0	91	0	1	0	1
9	070	1	0	1	0	1	0	1
10	0	1	0	1	0	1	0	1
15	0	1	0	1	0	1	0	1
20	0	1	0	1	0	1	0	1
25	0	1	0	1	0	1	0	1
30	0	1	0	1	0	1	0	1
35	0	1	0	1	0	1	0	1
40	0	1	0	1	0	1	0	1
45	0	1	0	1	0	1	0	1
50	0	1	0	1	0	1	0	1
60	0	1	0	1	0	1	0	1
70	0	1	0	1	0	1	0	1
80	0	1	0	1	0	1	0	1
90	0	1	0	1	0	1	0	1
10	0	1	0	1	0	1	0	1
150	0	1	0	1	0	1	0	1
200	0	1	0	1	0	1	0	1

0

GRAPH 1



GRAPH 2



3. CONCLUSIONS

Generally, any manufacturer would like to relax the conditions imposed on material, machines and men in a situation where there is a possibility of doing so, viz. U-L > 6σ.

In the process, one may ignore the effect it has on the lifetime of the product, in terms of the product reliability. This, in turn, leads to loss to the customer and to the manufacturer in terms of the number of complaints on warranty period. Hence, a proper lifetime distribution to be used under situation would be the compound lifetime distributions rather than the conventional one.

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