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N-FOLD GENERALIZED HESITANT IMPLICATIVE FILTERS OF HOOPS

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Abstract:-

To investigate implicative filters of hoops furthermore, we apply n-fold theory to (α, β)-hesitant fuzzy implicative filters and introduce the notion of n-fold (α,β)-hesitant fuzzy implicative filters, and obtain some conditions for a (α,β)-hesitant fuzzy filter to be a n-fold (α,β)-hesitant fuzzy implicative filter. We also study the preimage and image of n-fold (α, β)-hesitant fuzzy implicative filters.

Keywords: - *Hoop; (α, β)-hesitant fuzzy filter; n-fold (α,β)-hesitant fuzzy implicative filter*

1. INTRODUCTION

Hoop-algebras or briefly hoops as ordered commutative residuated integral monoids satisfying a further conditions, were introduced by Bosbach [1]. The study of hoops has experienced a tremendous growth and more and more algebraic properties have been investigated [2, 3, 4]. In studying hoops, filters play an important role in the logical point of view and various filters correspond to various sets of provable formulae. Kondo [5] considered that fundamental properties of filters in hoops and proved that any positive filter of a hoop is implicative and fantastic. To extend the research to filter theory of hoops, [6] gave the notions of some types of filters (positive) implicative filters, fantastic filters, associative filters) in pseudo hoop-algebras and investigated their properties. [7] introduced the notions of *n*-fold (positive) implicative filters,

There are many complicated problems in real life that involve uncertain data. The hesitant fuzzy set [8] is a very useful tool to deal with uncertainty in avoiding such issues in which each criterion can be described as a hesitant fuzzy element defined in terms of the opinions of experts and permits the membership having a set of possible values in decision making. In the same time, hesitant fuzzy set theory has been applied to investigate algebraic structures, such as MTL-algebras [9] and BCK/BCIalgebras [10]. Yang et al. put forward a new hesitant fuzzy filterł(α,β)-hesitant fuzzy filter, which is a generalization of hesitant fuzzy filters [11]. To extend the research of *alpha*,β-hesitant fuzzy filter, Yang further studied some characterizations of (α,β)-hesitant implicative fuzzy filters of hoops [12].

Considering that the notions of *n*-fold implicative filters [7] and (α,β)-hesitant fuzzy implicative filters [12], we present the notion of *n*-fold (α,β)-hesitant fuzzy implicative filters in hoops. Some characterizations of *n*-fold (α,β)-hesitant fuzzy implicative filters are discussed. The preimage and image of *n*-fold (α,β)-hesitant fuzzy implicative filters are also investigated.

2. Preliminaries

To facilitate our discussion, we first review some backgrounds of hoops and hesitant fuzzy sets.

An algebra $(A, \otimes, \rightarrow, 1)$ of type (2,2,0) is called a hoop if it satisfies the following conditions: for any $x,y,z \in A$,
(HP1) $(H, \otimes, 1)$ is a commutative monoid,
(HP2) $x \rightarrow x = 1$,
(HP3) $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$,
(HP4) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$.

In the following, unless mentioned otherwise, $(A, \otimes, \rightarrow, 1)$ will be a hoop, which will often be referred by its support set *A*.

The order relation "≤" on *A* is defined by $x \leq y$ if and only if $x \rightarrow y = 1$ for any $x,y \in A$. We put $x \wedge y = x \otimes (x \rightarrow y)$

$$x^n = \underbrace{x \otimes \cdots \otimes x}_{n \text{ times}} \text{ if } n > 0 \text{ and } x^0 = 1.$$

and denote

Proposition 2.1. [13, 14] *Let* $(A, \otimes, \rightarrow, 1)$ *be a hoop. Then the following assertions are valid: for any* $x,y,z \in A$,
(1) $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$,
(2) $x \otimes (x \rightarrow y) \leq y$, $x \otimes y \leq x \wedge y \leq x \rightarrow y$, $x \leq y \rightarrow x$,
(3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
(4) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,
(5) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(6) *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$ *and* $x \otimes z \leq y \otimes z$.

Let *F* a nonempty subset of *A*. *F* is called a filter if it satisfies: for any $x,y \in A$, (i) $x,y \in F$ implies $x \otimes y \in F$; (ii) $x \in F$ and $x \leq y$ imply $y \in F$. It is shown that a nonempty subset *F* of *A* is a filter if and only if for any $x,y \in A$, (i) $1 \in F$; (ii) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$. Moreover, a non-empty set *F* of *A* is called an implicative filter of *A* if it satisfies that $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$, for any $x,y,z \in A$ [7].

Definition 2.2. [7] *Let F be a subset of A and* $n \in N$. *Then F is called a n-fold implicative filter of A if it satisfies: for any* $x,y,z \in A$
(1) $1 \in F$,
(2) $x^n \rightarrow (y \rightarrow z) \in F$ *and* $x^n \rightarrow y \in F$ *imply* $x^n \rightarrow z \in F$.

Definition 2.3. [8] *Let E be a reference set. A hesitant fuzzy set* H *on E is defined in terms of a function h that when applied to E returns a subset of* [0,1], *i.e.,*

$$\mathrm{H} = \{(e,h(e)) | e \in E\},$$

where h(*e*) *is a set of some different values in* [0,1], *representing the possible membership degrees of the element* $e \in E$ *to* H.

For convenience, the hesitant fuzzy set H will often be referred to by its function *h*. In what follows, we take a hoop *A* as a reference set and $\emptyset \subseteq \alpha \subset \beta \subseteq [0,1]$.

Yang et al. [11] introduced following notations. Let *h* be hesitant fuzzy set of *A*. For any $x,y \in A$,
(1) $h(x) \subseteq_\beta^\alpha h(y)$ means that $h(x) \cap \beta \subseteq h(y) \cup \alpha$,
(2) $h(x) =_\beta^\alpha h(y)$ means that $(h(x) \cap \beta) \cup \alpha = (h(x) \cap \beta) \cup \alpha$.

It is easy to verify that $h(x) =_\beta^\alpha h(y)$ iff $h(x) \subseteq_\beta^\alpha h(y)$ and $h(y) \subseteq_\beta^\alpha h(x)$. And so $h(x) \subseteq_\beta^\alpha h(y)$ iff $(h(x) \cap \beta) \cup \alpha \subseteq (h(x) \cap \beta) \cup \alpha$.

Lemma 2.4. [11] *Let h be a hesitant fuzzy set of A. Then for any x,y ∈ A,*
(1) $h(x) \subseteq_\beta^\alpha h(y)$ *and* $h(y) \subseteq_\beta^\alpha h(z)$ *imply* $h(x) \subseteq_\beta^\alpha h(z)$,
(2) $h(x) \subseteq_\beta^\alpha h(y)$ *implies* $h(x) =_\beta^\alpha h(x) \cap h(y)$,
(3) $h(x) \subseteq_\beta^\alpha h(y)$ *implies* $h(x) \cap h(z) \subseteq_\beta^\alpha h(y) \cap h(z)$,
(4) $h(x) \subseteq_\beta^\alpha h(y)$ *and* $h(x) \subseteq_\beta^\alpha h(z)$ *imply* $h(x) \subseteq_\beta^\alpha h(y) \cap h(z)$.

A hesitant fuzzy set h of A is called a (α,β)-hesitant fuzzy filter if if it satisfies: for any $x,y \in A$, (i) $h(x) \cap h(y) \subseteq_\beta^\alpha h(x \otimes y)$, (ii) $x \leq y$ implies that $h(x) \subseteq_\beta^\alpha h(y)$. It has been proved that a hesitant fuzzy set h of A is a (α,β)-hesitant fuzzy filter if and only if (i) $h(x) \subseteq_\beta^\alpha h(1)$, (ii) $h(x) \cap h(x \to y) \subseteq_\beta^\alpha h(y)$ for any $x,y \in A$ [11].

Definition 2.5. [12] *A hesitant fuzzy set h of A is called a* (α,β)*-hesitant fuzzy filter if if it satisfies: for any x,y* ∈ *A,*
(1) $h(x) \subseteq_\beta^\alpha h(1)$,
(2) $h(x \to y) \cap h((x \to y) \to z) \subseteq_\beta^\alpha h(x \to z)$.

Proposition 2.6. [12] *Every* (α,β)*-hesitant fuzzy implicative filter of a hoop is a* (α,β)*-hesitant fuzzy filter.*

Definition 2.7. *Let* A_1 *and* A_2 *be two hoops. A function* $f : A_1 \to A_2$ *is called a hoop-homomorphism if*
(1) $f(1) = 1$,
(2) $f(a \otimes b) = f(a) \otimes f(b)$, (3) $f(a \to b) = f(a) \to f(b)$, *for any* $a,b \in A_1$.

3. *n*-fold (α, β)-hesitant fuzzy implicative filters

In the section, we give the notion of *n*-fold (α,β)-hesitant fuzzy implicative filters of hoops, and present some characterizations of it.

Definition 3.1. *Let h be a hesitant fuzzy set of A and n* ∈ *N. Then h is called a n-fold* (α,β)*-hesitant fuzzy implicative filter of A if if it satisfies: for any x,y* ∈ *A,*
(1) $h(x) \subseteq_\beta^\alpha h(1)$,
(2) $h(x^n \to y) \cap h((x^n \to (y \to z)) \subseteq_\beta^\alpha h(x^n \to z)$.

Remark 3.2. (1) *Notice that* 1*-fold* (α,β)*-hesitant fuzzy implicative filter of a hoop is a* (α,β)*-hesitant fuzzy implicative filter.*
(2) *The notion of n-fold* (α,β)*-hesitant fuzzy filters of a hoop generalizes the notion of* (α,β)*-hesitant fuzzy implicative filters.*

By taking $x = 1$ in Definition 3.1, it is to see that every *n*-fold (α,β)-hesitant fuzzy filter of a hoop is a (α,β)-hesitant fuzzy filter.

Example 3.3. *Let* $A = \{0,a,b,1\}$ *be a set with Cayley tables as follows.*

⊗	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

→	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(H, \otimes, \to, 1)$ *is a hoop. Let* $\alpha = \{0.2, 0.3\}$ *and* $\beta = \{0.2, 0.3, 0.6\}$. *Define a hesitant fuzzy set h of A as*

$$h(t) = \begin{cases} \{0.2, 0.5\}, & t = 0, \\ \{0.1, 0.3, 0.5\}, & t = a, \\ \{0.2, 0.5\}, & t = b, \\ \{0., 0.8\}, & t = 1. \end{cases}$$

Routine calculations show that h is a 3*-fold* (α,β)*-hesitant fuzzy filter of A.*

Theorem 3.4. *Let h be a* (α,β)*-hesitant fuzzy filter of A of A and n* ∈ *N. Then the following are equivalent: for any x,y,z* ∈ *A,*
(1) *h is a n-fold* (α,β)*-hesitant fuzzy implicative filter of A,*
(2) $h(x^{n+1} \to y) \subseteq_\beta^\alpha h(x^n \to y)$,
(3) $h(x^n \to x^{2n}) =_\beta^\alpha h(1)$,
(4) $h(x^n \to (y \to z)) \subseteq_\beta^\alpha h((x^n \to y) \to (x^n \to z))$.

Proof. (1) ⇒ (2) Since $x^n \to x = 1$, then $h(x^{n+1} \to y) =^{\alpha}_{\beta} h(x^{n+1} \to y) y)) \cap h(x^n \to x) \subseteq^{\alpha}_{\beta} h(x^n \to y)$ for any $x,y \in A$, thus (2) holds.
(2) ⇒ (3) The proof is by induction on n. Suppose that (2) holds.

Firstly, for $n = 1$, $h(1) = h(x^{1+1} \to x^2) \subseteq^{\alpha}_{\beta} h(x \to x^2)$, we have $h(x \to x^2) =^{\alpha}_{\beta} h(1)$.
Secondly, for $n = 2$, we get that

$$\begin{aligned} h(1) &= h(x^3 \to (x^1 \to x^4)) \\ &\subseteq^{\alpha}_{\beta} h(x^2 \to (x \to x^4)) \\ &= h(x^3 \to x^4) \\ &\subseteq^{\alpha}_{\beta} h(x^2 \to x4), \end{aligned}$$

and so $h(x^2 \to x^4) =^{\alpha}_{\beta} h(1)$.
Finally, for $n > 2$, since $x^{n+1} \to (x^{n-1} \to x^{2n}) = 1$, then

$$\begin{aligned} h(1) &= h(x^{n+1} \to (x^{n-1} \to x^{2n})) \\ &\subseteq^{\alpha}_{\beta} h(x^n \to (x^{n-1} \to x^{2n})) \\ &= h(x^{n-1} \to (x^n \to x^{2n})), \end{aligned}$$

and therefore $h(x^{n-1} \to (x^n \to x^{2n})) =^{\alpha}_{\beta} h(1)$. By using the hypothesis n times, then we get $h(x^{n-n} \to (x^n \to x^{2n})) =^{\alpha}_{\beta} h(1)$, thus $h(x^n \to x^{2n}) =^{\alpha}_{\beta} h(1)$.

(3) ⇒ (4) Noticing that $y \to z \le (x^n \to y) \to (x^n \to z)$ for any $x,y,z \in A$, we get that

$$\begin{aligned} x^n \to (y \to z) &\le x^n \to ((x^n \to y) \to (x^n \to z)) \\ &= x^n \to (x^n \to ((x^n \to y) \to z)) \\ &= x^{2n} \to ((x^n \to y) \to z) \\ &\le (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z)). \end{aligned}$$

Consider that h is a (α,β)-hesitant fuzzy filter and $h(x^n \to x^{2n}) =^{\alpha}_{\beta} h(1)$, we get that

$$\begin{aligned} h(x^n \to (y \to z)) &\subseteq^{\alpha}_{\beta} h((x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))) \\ &=^{\alpha}_{\beta} h(x^n \to x^{2n}) \cap h((x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))) \\ &\subseteq^{\alpha}_{\beta} h((x^n \to y) \to (x^n \to z)), \end{aligned}$$

this means that $h(x^n \to (y \to z)) \subseteq^{\alpha}_{\beta} h((x^n \to y) \to (x^n \to z))$.

(4) ⇒ (1) By hypothesis that h is a (α,β)-hesitant fuzzy filter, we obtain that

$$\begin{aligned} h(x^n \to y) \cap h(x^n \to (y \to z)) &\subseteq^{\alpha}_{\beta} h(x^n \to y) \cap h((x^n \to y) \to (x^n \to z)) \\ &\subseteq^{\alpha}_{\beta} h(x^n \to z), \end{aligned}$$

therefore h is a n-fold (α,β)-hesitant fuzzy implicative filter of A.

Theorem 3.5. *Let h be a* (α,β)*-hesitant fuzzy filter of A and n* ∈ *N. Then h is a n-fold* (α,β)*-hesitant fuzzy implicative filter if and only if* $h(x^{2n} \to y) \subseteq^{\alpha}_{\beta} h(x^n \to y)$ *for any* $x,y \in A$.
Proof. Assume that h is a n-fold (α,β)-hesitant fuzzy implicative filter, then taking $y = x^n$ and $z = y$ in Theorem 3.4 (4), we have $h(x^{2n} \to y) = h(x^n \to (x^n \to y)) \subseteq^{\alpha}_{\beta} h((x^n \to x^n) \to (x^n \to y)) = h(x^n \to y)$, that is, $h(x^{2n} \to y) \subseteq^{\alpha}_{\beta} h(x^n \to y)$.
Conversely, it follows that $h(1) = h(x^{2n} \to x^{2n}) \subseteq^{\alpha}_{\beta} h(x^n \to x^{2n})$, and so $h(1) =_{\beta}^{\alpha} h(x^n \to x^{2n})$.
According to Theorem 3.4, we get that h is a n-fold (α,β)-hesitant fuzzy implicative filter of A.

Proposition 3.6. *If h is a n-fold* (α,β)*-hesitant fuzzy implicative filter of A, then for any* $x,y,z \in A$,
(1) $h(x^{n+1} \to y) =^{\alpha}_{\beta} h(x^n \to y)$,
(2) $h(x^n \to (y \to z)) =^{\alpha}_{\beta} h((x^n \to y) \to (x^n \to z))$,
(3) $h(x^{2n} \to y) =^{\alpha}_{\beta} h(x^n \to y)$.

Proof. (1) Since h is a n-fold (α,β)-hesitant fuzzy implicative filter of A, then $h(x^{n+1} \to y) \subseteq^{\alpha}_{\beta} h(x^n \to y)$ by Theorem 3.4. As for the reverse inclusion, from $x^n \to y \le x^{n+1} \to y$, we have $h(x^n \to y) \subseteq^{\alpha}_{\beta} h(x^{n+1} \to y)$. Thus $h(x^n \to y) =^{\alpha}_{\beta} h(x^{n+1} \to y)$.

(2) Notice that h is a n-fold (α,β)-hesitant fuzzy implicative filter of A, we get that $h(x^n \to (y \to z)) \subseteq^{\alpha}_{\beta} h((x^n \to y) \to (x^n \to z))$ by Theorem 3.4. For the converse, since $(x^n \to y) \to (x^n \to z) \le x^n \to (y \to z)$ by Proposition 2.1 (6), we have $h((x^n \to y) \to (x^n \to z)) \subseteq^{\alpha}_{\beta} h(x^n \to (y \to z))$, thus $h(x^n \to (y \to z)) =^{\alpha}_{\beta} h((x^n \to y) \to (x^n \to z))$.

(3) According to Theorem 3.5, we get that $h(x^{2n} \to y) \subseteq_\beta^\alpha h(x^n \to y)$ for any $x,y \in A$. Moreover, since $x^n \to y \le x^{2n} \to y$, then $h(x^n \to y) \subseteq_\beta^\alpha h(x^{2n} \to y)$, hence we conclude $h(x^{2n} \to y) =_\beta^\alpha h(x^n \to y)$.

Theorem 3.7. *Let h be a* (α,β)*-hesitant fuzzy filter of A and* $n \in N$*. Then h is a n-fold* (α,β)*-hesitant fuzzy implicative filter if and only if* $h(z \to (x^{2n} \to y)) \cap h(z) \subseteq_\beta^\alpha h(x^n \to y)$ *for any* $x,y,z \in A$.

Proof. Suppose that h is a n-fold (α,β)-hesitant fuzzy implicative filter, then $h(z \to (x^{2n} \to y)) \cap h(z) \subseteq_\beta^\alpha h(x^{2n} \to y) \subseteq_\beta^\alpha h(x^n \to y)$ by Theorem 3.5, and so, $h(z \to (x^{2n} \to y)) \cap h(z) \subseteq_\beta^\alpha h(x^n \to y)$.

Conversely, taking $z = 1$, we get $h(1 \to (x^{2n} \to y)) \cap h(1) =_\beta^\alpha h(x^{2n} \to y) \subseteq_\beta^\alpha h(x^n \to y)$. According to Theorem 3.5, we get that h is a n-fold (α,β)-hesitant fuzzy implicative filter of A.

Lemma 3.8. *Every n-fold* (α,β)*-hesitant fuzzy implicative filter of A is a* $(n + 1)$*-fold* (α,β)*-hesitant fuzzy implicative filter.*

Proof. Let h is a n-fold (α,β)-hesitant fuzzy implicative filter of A. Then $h(x^{n+2} \to y) = h(x^{n+1} \to (x \to y)) \subseteq_\beta^\alpha h(x^n \to (x \to y)) = h(x^{n+1} \to y)$, that is, $h(x^{n+2} \to y) \subseteq_\beta^\alpha h(x^{n+1} \to y)$. Using Theorem 3.5, we have that h is a $(n + 1)$-fold (α,β)-hesitant fuzzy implicative filter.

Proposition 3.9. *If h is a n-fold* (α,β)*-hesitant fuzzy implicative filter of A, then h is* $(n + k)$*-fold* (α,β)*hesitant fuzzy implicative filters.*

The following result shows that the relationship between n-fold (α,β)-hesitant fuzzy implicative filters and n-fold implicative filter.

Proposition 3.10. *If h is a n-fold* (α,β)*-hesitant fuzzy implicative filter of A, then the set* $ker(h) := \{x \in A | h(x) =^\alpha_\beta h(1)\}$ *is a n-fold implicative filter of A.*

Proof. Obviously, $1 \in ker(h)$. For any $x^n \to (y \to z) \in ker(h)$ and $x^n \to y \in ker(h)$, then $h(x^n \to (y \to z)) =^\alpha_\beta h(1)$ and $h(x^n \to y) =_\beta^\alpha h(1)$. Notice that h is a n-fold (α,β)-hesitant fuzzy implicative filter of A, we obtain that $h(1) =_\beta^\alpha h(x^n \to (y \to z)) \cap h(x^n \to y) \subseteq_\beta^\alpha h(x^n \to z) \le$, and so $h(x^n \to z) =_\beta^\alpha h(1)$, it follows that $x^n \to z \in ker(h)$. Thus $ker(h)$ is a n-fold implicative filter of A.

Proposition 3.11. *Let* h_1,h_2 *be two hesitant fuzzy sets of A with* $h_1(1) =^\alpha_\beta h_2(1)$ *and* $h_1 \vee^\alpha_\beta h_2$*, that is,* $h_1(x) \subseteq_\beta^\alpha h_2(x)$ *for any* $x \in A$*. If* h_1 *is a n-fold* (α,β)*-hesitant fuzzy implicative filter of A, then* h_2 *is also a n-fold* (α,β)*-hesitant fuzzy implicative filter of A.*

Proof. Notice that h_1 is a n-fold (α,β)-hesitant fuzzy implicative filter of A, then $h_1(1) =^\alpha_\beta h_1(x^n \to x^{2n}) \subseteq_\beta^\alpha h_2(x^n \to x^{2n})$ for any $x \in A$. It follows that $h_2(x^n \to x^{2n}) =_\beta^\alpha h_2(1)$, hence h_2 is also a n-fold (α,β)-hesitant fuzzy implicative filter of A.

Definition 3.12. *Let* A_1*,* A_2 *be two hoops,* $f : A_1 \to A_2$ *be a map, and* h_1,h_2 *be hesitant fuzzy sets of* A_1 *and* A_2*, respectively. Then*

(1) *the preimage* $f^{-1}(h_2)$ *of* h_2 *under f is defined as* $f^{-1}(h_2)(x) =^\alpha_\beta h_2(f(x))$*, for any* $x \in A_1$*,*

(2) *the image* $f(h_1)$ *of* h_1 *under f is defined as*

$$f(h_1)(y) =_\beta^\alpha \begin{cases} \bigcup\{h_1(x) | f(x) = y\}, & f^{-1}(y) \neq \emptyset, \\ \emptyset, & otherwise. \end{cases}$$

Theorem 3.13. *Let* h_1,h_2 *be n-fold* (α,β)*-hesitant fuzzy implicative filters of* A_1 *and* A_2*, respectively.*

(1) *If* $f : A_1 \to A_2$ *is a hoop-homomorphism, then the preimage* $f^{-1}(h_2)$ *is a n-fold* (α,β)*-hesitant fuzzy implicative filter of* A_1*.*

(2) *If f is a hoop-epimorphism, then the image* $f(h_1)$ *is a n-fold* (α,β)*-hesitant fuzzy implicative filter of* A_2*.*

Proof. It is easy to prove that $f^{-1}(h_2)$ and $f(h_1)$ are (α,β)-hesitant fuzzy filters of A_1 and A_2, respectively.

(1) $f^{-1}(h_2)(x^n \to x^{2n}) =^\alpha_\beta h_2(f(x^n \to x^{2n})) = h_2(f(x)^n \to f(x)^{2n}) =^\alpha_\beta h_2(1)$, hence $f^{-1}(h_2)$ is a n-fold (α,β)-hesitant fuzzy implicative filter of A_1.

(2) Note that h_1 is a n-fold implicative pseudo valuations on H_1 and f is a hoop-epimorphism, For any $y \in H_2$, then there exists $x \in H_1$ such that $f(x) = y$. It follows that

$$\begin{aligned} f(h_1)(y^n \to y^{2n}) &=_\beta^\alpha \bigcup\{h_1(z) | f(z) = y^n \to y^{2n}, z \in A_1\} \\ &=_\beta^\alpha \bigcup\{h_1(z) | f(z) = f(x)^n \to f(x)^{2n}, z \in A_1\} \\ &=_\beta^\alpha \bigcup\{h_1(z) | f(z) = f(x^n \to x^{2n}), z \in A_1\} \\ &=_\beta^\alpha h_1(1), \end{aligned}$$

and so $f(h_1)$ is a n-fold (α,β)-hesitant fuzzy implicative filter of A_2.

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