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ON THE CONVERGENCE OF SOME EIGENFUNCTION EXPANSIONS

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Abstract:-
In this paper we presented an extension of previous results given in the papers [15], [17],[19], the main idea of the proofs is: we write down the difference of the trigonometric kernel of the general expansion considered, and we have to estimate the resulting infinite sums. For the terms of these sums we used sharper and different estimates than in the previous papers in the literature (the most exact estimates where given by V.A.Il'n, I. Joó and V.Komornik).

Key words:- *Series Expansions, Eigenfunction Expansions, Riesz bases.*

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1. INTRODUCTION

Many central problems of spectral theory of linear operators are concentrated around the problem of eigenfunction expansions. From one hand it accumulates questions of eigenvalues and eigenfunctions asymptotic, from the other it connects mathematics with many physical problems of string and membrane vibrations, of quantum mechanics and so on. The difference of eigenfunction expansions converges to zero in any interior point of the main interval, was called equiconvergence and it makes possible to reduce numerous questions of point and uniform convergence to those of some model, usually, trigonometric system.

The equiconvergence theorems are very useful in the spectral investigation of differential operators, because many results known for the most special operators may be transferred by their applications to more general ones.
One of the first results of this type was proved by A.Haar [1] in 1910-1911, after by N. Wiener and J.L. Walsh in 1921[2].

In order to investigate eigenfunctions expansions three estimates are essential:

1) Upper estimate of one eigenfunction,
2) Upper estimate of sum of squares of eigenfunctios,
3) Titchmarsh type mean value formula.

In 1977 a new and fruitful method was developed by V.A.Il'n (cf.[3],[4]). His method works for the first and second estimates in the case of ordinary differential operator of second order (Schrödinger operator) with $q \in L^{2.}$ For n-th order differential operator (with smooth coefficients and using the fundamental solution) only the first estimate was proved, and he assumed that the second was fulfilled.
I.Joó gave for both estimates a new procedure (cf.[5],[6],[7]) without using the fundamental solution in case $q\in L^1_{loc}$. These results led I.Joó and V.Komornik [8] to very general equiconvergence theorem for the Schrödinger operator.
This theorem concerns expansions by Riesz bases formed by eigenfunctions of higher order Of the Schršdinger operator.The existence of Riesz basis consisting of eigenfunctions of higher order was proved by V.P.Mikhailov [9] and G.m.Keselman [10].

As another illustration we can mention that there exist Riesz bases $\left\{c_n e^{i\lambda_n t}\right\}_{n\in\square}$ with $\sup \operatorname{Im} \lambda = +\infty$; the construction is described in M.Horváth dissertation [11].
Joó's method made it possible to extend 1) and 2) for the differential operators of n-th order with any (smooth or not) coefficients. This was done by V.Komornik in large number of papers (cf.[12],[13],[14]),using and developing some ideas and results of Joó-Komornik[8] and Joó's papers ([5],[6],[7]).The results of Komornik are new also when the coefficients of the differential operator are smooth.
A generalization of the mentioned paper Joó and Komornik [8] is given in Komornik [15] (for higher order differential operators) and it is based on the results of ([12],[13],[14]).For this Komornik had to extend also the Titchmarsh formula [14] and he need the explicit formulas for its coefficients given by Joó [16].
Recently a general equiconvergence theorem was published in [17] by Horváth, Joó and Komornik for the one dimentional Schrödinger operator without any restriction of the distribution of the eigenvalues on the complex plane, generalizing some known classical results of the field. The proof uses some estimates of [18] given by Joó.

2. The method:
The main idea of the proofs is: we write down the difference of the trigonometric kernel of the general expansion considered, and we have to estimate the resulting infinite sums.For the terms of these sums we used sharper and different estimates than in the previous papers in the literature (the most exact estimates where given by V.A.Il'n, I. Joó and V.Komornik). Now we give the reason why our proof is harder and longer than Komornik's proof: $f\in L^{2,}(u_n)$ is not (known) Riesz bases:

$$\int_G f(y)(w^R(|x-y|,\mu)-\sum_{\sqrt{\lambda_n}<\mu} u_n(x)v_n(y))dy$$

$$=\int(\sum_{n=1} c(\mu,\lambda_n).u_n(x)v_n(y)f(y))dy$$

$$=\sum_{n=1}^{\infty} c(\mu,\lambda_n).\underbrace{u_n(x)}_{\le\|u_n\|_{L^2(G)}}.\int_G \underbrace{v_n(y)f(y))}_{\le\|v_n\|_{L^2(G)}\cdot\|f\|_{L^2(G)}} dy$$

$$\le\|f\|_{L^2(G)}.\sum_{n=1}^{\infty} c(\mu,\lambda_n).\|u_n\|_{L^2(G)}.\|v_n\|_{L^2(G)}$$

$$\le c\|f\|_{L^2(G)}.\sum_{k=1}^{\infty}\left|c(\mu,k^2)\right|.\sum_{k\le\sqrt{\lambda_n}\le k+1} 1\le c(x)$$

$c(x)$ independent on μ . (we proved this).

But if $f \in L^2$, and (u_n) is Riesz bases then : $\sum |f_n|^2 < \infty$, and we can estimate as follows:

$$\int_G \Big(\sum_{n=1}^{\infty} c(\mu,\lambda_n).u_n(x)v(y)\Big) f(y)dy = \sum_{n=1}^{\infty} c(\mu,\lambda_n).u_n(x).\int v(y)f(y)dy$$

$$\le \Big(\sum_{n=1}^{\infty}\Big|c(\mu,\lambda_n)\underbrace{u_n(x)}_{\le c}\Big|^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{1/2}$$

$$\le c\sum_{n=1}^{\infty}|c(\mu,\lambda_n)|^2 = c\sum_{k=1}|c(\mu,k^2)|^2 . \sum_{k\le\sqrt{\lambda_n}\le k+1} 1 \le c(x)$$

$c(x)$ independent on μ . (Komornik proved this).

3. The main results:

In our paper [20] which generalized [17] by using the estimates developed by Komornik in his papers ([14], [15]), and by Joó in his paper [18].

Let G be an open interval (finite or infinite) on the real line, n a natural number, $q_s \in L^1_{loc}(G)\,(s=2,3,\dots,n)$ a complex functions and consider the differential operator:

$$\boxed{Lu := u^{(n)} + q_2(x)u^{(n-2)} + \dots + q_n(x)u} \quad (n \ge 2)$$

Defined on $H\,G_{loc}{}^n(\ )$.(Recall that, by definition, $H\,G_{loc}{}^n(\ )$ is the set of all complex functions $v \in L^2{}_{loc}(G)$ having distributional derivatives in $L^2{}_{loc}(G)$ of order up to k).

Given a complex numberλ, the function $u\ G: \to \square\ u = 0$ is called an eigenfunction of order -1 of the operator L with eigenvalueλ. Furthermore , a function $u\ G: \to \square\ u = 0$, is called An eigenfunction of order k(k=0,1, …)of the operator L with the eigenvalues λ if the function $u^* = L\,u - \lambda u$ is an eigenfunction of order (k-1) with the same eigenvalues λ.

Let us now we given a complete and minimal system $(u_\alpha) \subset L^2(G)$of eigen functions of the operator L, denote by λ_α (resp. o_α)the eigenvalue (resp. order) of u_α and assume:

1) $\sup_\alpha o_\alpha < \infty$,

2) in case $o_\alpha > 0$, $\lambda_\alpha u_\alpha - Lu_\alpha = u_{\alpha-1}$

We introduce some notations:

Index the nth roots of λ_α such that $\operatorname{Re}\mu_{1,\alpha} \ge \dots \ge \operatorname{Re}\mu_{n,\alpha}$

and put $\mu_\alpha := \mu_{m,\alpha}$. $\operatorname{Im}\mu_{j,\alpha} > \operatorname{Im}\mu_{j+1,\alpha}$, *in case* $\operatorname{Re}\mu_{j,\alpha} = \operatorname{Re}\mu_{j+1,\alpha}$

and put

$$\mu_\alpha := \mu_{m,\alpha} \ , \ \rho_\alpha := |\operatorname{Re}\mu_\alpha| \ , \ \nu_\alpha := |\operatorname{Im}\mu_\alpha| \quad where \ \ m = \left[\frac{n+1}{2}\right];$$

$$\delta(\nu,\nu_\alpha) = \begin{cases} 1, & \nu > \nu_\alpha \\ 1/2, & \nu = \nu_\alpha \\ 0, & \nu < \nu_\alpha \end{cases}$$

$$W_R(t) := \begin{cases} \dfrac{\sin\nu(x-t)}{\pi(x-t)}, & |x-t| \le R \\ o, & |x-t| > R \end{cases}$$

where $x \in K, K$ is an arbitrary fixed compact interval $K \subset G$ and $R \in (0, dist(K,\partial G))$;

$$D_{R_o} f := \frac{2}{R_o}\int_{\frac{R_o}{2}}^{R_o} f(R)dR, \quad 0 < R_o < dist(K,\partial G); \quad W(t) := D_{R_o}(W_R)$$

$$\sigma_\nu(f,x) := \sum_{\nu_\alpha<\nu}(f,\nu_\alpha)u_\alpha(x) + \sum_{\nu_\alpha=\nu}^{*} c_\alpha(f,\nu_\alpha)u_\alpha(x)$$

where c_α are arbitrary constants, $|c_\alpha| \le C$, *and* $\sum^*$ denotes the sum for any subset of $\{\alpha : \nu_\alpha = \nu\}$, $f \in L^2(G), \nu > 0, x \in G, (v_\alpha)$ is the dual system of (u_α)

$(i.e.(v_\alpha) \subset L^2(G)$ *and* $\langle v_\alpha, u_j \rangle = \delta_{k,j})$;

$$S_\nu(f,x) := \int_{x-R}^{x+R} \frac{\sin \nu(y-x)}{\pi(y-x)} f(y)dy,$$

where $f \in L^2(G), \nu > 0, x \pm R \in G$; $K_b := \{x \in G : dist(x,K) \le b\}$

where $K \subset G$ is a compact interval and $0 < b < dist(K, \partial G)$. We prove the following :

Theorem 1. Assume that $(1),(2)$, $q \equiv 0, u_\alpha^* \equiv 0$ *and* $\sup_{t>0} \sum_{t \le \nu_\alpha \le t+1} 1 < \infty$ are fulfilled.
Then the following three statements are equivalent:

(a) For any compact interval $K \subset G$

$$\sup_\alpha \|v_\alpha\|_{L^2(G)} \cdot \|u_\alpha\|_{L^2(G)} < \infty$$

(b) For any compact interval $K \subset G$ and any subsume $\sum^*$

$$\lim_{\nu\to\infty} \sup_{x \in K} |S_\nu(f,x) - \sigma_\alpha(f,x)| = 0$$

for every $f \in L^2(G)$ *and every* $0 < R < dist(K, \partial G)$.

(c) For any compact interval $K \subset G$ and any subsume $\sum^*$

$$\lim_{\nu\to\infty} \|f - \sigma_\nu(f)\|_{L^2(G)} = 0 \quad \text{for every } f \in L^2(G).$$

This is the results of our paper [20], and now we are working to eliminate the conditions $q_2 = 0$ *and* $u^* = 0$.
For n =4 this was proved by the author in [21] earlier.
And in [24] we gave the results which continued some recent investigations of Joó in [21].
We will give now some notations and theorem of our results in [24]:
Let G be an arbitrary (finite or infinite) open interval on the real line, q q, $\hat{} \in L^1_{loc}(G)$ be arbitrary complex functions.
Let $(u_k)(resp.(\hat{u}_k))$ be a Riesz-basis in L^2 (G) consisting of eigen funcions of the operator $Lu = -u'' + qu(resp. \hat{L}u = -u'' + \hat{q}u)$ and having the following properties:

$1) \sup o_k < \infty \quad \sup \hat{o}_k < \infty$

$2)$ *in case* $o_k > 0 \quad (resp.\ \hat{o}_k > 0)$

$\lambda_k u_k - Lu_k = u_{k-1}\ .(resp. \hat{\lambda}_k \hat{u}_k - \hat{L}\hat{u}_k = \hat{u}_{k-1}),$

Where λ_k *and* $o_k(resp. \lambda_k$ *and* $\hat{o}_k)$ are the eigenvalue and the order 0f $(u_k)(resp.(\hat{u}_k))$
Now let us introduce some notations:

$$(3)\ R_\mu(f,x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<2\mu} \langle f, v_k \rangle u_k(x)(1 - \frac{\mu_k}{2\mu}), \quad (\mu_k = \sqrt{\lambda_k}),$$

$$\hat{R}_\mu(f,x) := \sum_{|\mathrm{Re}\sqrt{\hat{\lambda}_k}|<2\mu} \langle f, \hat{v}_k \rangle \hat{u}_k(x)(1 - \frac{\hat{\mu}_k}{2\mu}), \quad (\hat{\mu}_k = \sqrt{\hat{\lambda}_k});$$

$(f \in L^2(G), x \in G, \mu > 0)$, *where* $(v_k)(resp.(\hat{v}_k))$ *is the dual system of*

$(u_k)(resp.(\hat{u}_k)), i.e. (v_k), (\hat{v}_k) \subset L^2(G)$ *and* $\langle v_k, u_j \rangle = \langle \hat{v}_k, \hat{u}_j \rangle = \delta_{k,j}$.

The following result holds:

Theorem 2. Given any compact interval $K\subset G$ for all $f\in L^2(G)$ (is finite or infinite)

$$\lim_{\mu\to\infty} \underset{x\in K}{sub} \left| R_\mu(f,x) - \hat{R}_\mu(f,x) \right| = 0$$

Also we have proved the same theorem for $f \in L^1(G)$ (G is finite or infinite)

Remark. If we modify the definition of R_μ,

$$R^*_\mu(f,x) := \sum_{\left|\mathrm{Re}\sqrt{\lambda_k}\right|<2\mu} \langle f, v_k \rangle u_k(x)\left(1 - \frac{\rho_k}{2\mu}\right), \quad (\rho_k = \sqrt{\lambda_k}),$$

The Theorem 2 remains true.

After that we investigate a special case.

Denote $$G = (0,+\infty), u_k(x) := \sqrt{2}x^{\alpha+\frac{1}{2}} e^{-x^2/2} l_k^{(\alpha)}(x^2),$$

is named the. Lagurre polynomial we proved the following theoram

Theorem 3

$$q(x) := x^2 - 2\alpha - 2 + \frac{2}{x^2}, \ \breve{\lambda}_k := 4k, \ where\, \alpha \geq -\frac{1}{2}, \quad l_k^{(\alpha)}(x)$$

If $f \in L^1(G), \ f'(t)(1+t^2) \in L^1(G), \lim_{+\infty} f = 0,$ then for any compact interval $K \subset G$ and for any sufficient small $R > 0$ we have

$$\sup_{x\in K} \left| F_\mu(f,x) - R_\mu(f,x) \right| = O\left(\frac{1}{\mu}\right),$$ where for

$$f \in L^2(G), \mu > 0\, and\, x \pm R \in G, define$$

$$(4) \quad F_\mu(f,x) = F_\mu(f,x,R) := \frac{1}{\mu\pi} \int_{x-R}^{x+R} \left(\frac{\sin \mu(y-x)}{y-x} \right) f(y)dy \cdot$$

The main idea of the proofs is: we write down the difference of the trigonometric kernel and the kernel of the general expansion considered, and we have to estimate the resulting infinite sums. For the terms of these sums we used sharper and different estimate than in the previous papers in the literature (the most exact estimates were given by V.A.Il'n , I.Joó and V.Komornik).

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