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A NOTE ON SOME INTERESTING THEOREMS OF THE MULATU NUMBERS

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Abstract:-
The Mulatu numbers were studied [1] and [2]. The numbers are sequences of numbers of the form: 4, 5,6,11,17,28,45...
The numbers have wonderful and amazing properties and patterns.
In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ M_{n-1} + M_{n-2} & \text{if } n > 1. \end{cases}$$

In [1] and [2] some properties and patterns of the numbers were considered. In this paper, we investigate additional properties and patterns of these fascinating numbers. Many beautiful mathematical identities involving the Mulatu numbers in relation with the Fibonacci numbers and the Lucas numbers will be more explored.

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Key Words:-*Mulatu numbers, Mulatu sequences, Fibonacci numbers, Lucas numbers, Fibonacci sequences, and Lucas sequences.*

INTRODUCTION AND BACKGROUND.

As given in [1], the Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, Professor of Mathematics at Savannah State University, Savannah, Georgia, and USA. The Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.
In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented. Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu (M_n), Fibonacci (F_n) and Lucas (L_n) Numbers

(Tables 1 & 2)

Table 1

n:	0	1	2	3	4	5	6	7	8	9	10	11
M_n:	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	12	13	14	15	16	17	18	19	20
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1: Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.
The following well-known identities of Mulatu numbers [1], Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $M_n = L_n + 2F_{n-1}$.

(4) $F_{2n} = F_n L_n$

(5) $5\,F^2{}_n - L^{2_n} = 4(-1)^{n+1}$

(6) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(7) $L_{n+1} = L_n + L_{n-1}$

(8) $F_{n+k} = F_{n-1}F_k + F_n F_{k+1}$

(9) $M_{-n} = (-1)^n M_n$

(10) $L_{n+m} = \dfrac{5F_n F_m + L_n L_m}{2}$

The Main Results.

Theorem 1. **Some Divisibility Properties of M.**

(a) If M_n is divisible by 2, then $M^2{}_{n+1} - M^2{}_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3{}_{n+1} - M^3{}_{n-1}$ is divisible by 9.

Proof: Note that: Using $M_{n+1} = (M_n + M_{n-1})$, we have:

(a) $M^2{}_{n+1} - M^2{}_{n-1}$
$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n\,(M_n + M_{n-1} + M_{n-1}) = M^2{}_n + 2M_nM_{n-1}$.

Now it is easy to see that if M_n is divisible by 2, then $M^2{}_{n+1}$ -$M^2{}_{n-1}$ is divisible by 4

(b) $M^3{}_{n+1} - M^3{}_{n-1} = (M_{n+1} - M_{n-1})(M^2{}_{n+1} + M_nM_{n-1} + M^2{}_{n-1})$

$= M_n\,(M^2{}_{n+1} + M_{n+1}M_{n-1} + M^2{}_{n-1})$

$= M_n((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2{}_{n-1})$

$= M_n(M^2{}_n + 3M_nM_{n-1} + 3M^2{}_{n-1})$

$= M^3{}_n + 3M^2{}_nM_{n-1} + 3M_nM^3{}_{n-1}$

Hence M_n is divisible by 3 $\Rightarrow M^3{}_{n+1}$ -$M^3{}_{n-1}$ is divisible by 9.

Remark1: Can we generalize Theorem 1 for n= 4, 5,6… ? We are still working on it.

Theorem 2. **The addition formula *for* Mulatu numbers**.

$$M_{n+k} = F_{n-1}M_k + F_nM_{k+1}$$

Proof: By Theorem 8[1] we have,

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}$$

Hence it follows that

$$M_{n+k} = F_{n+k-3} + F_{n+k-1} + M_{n+k+2}\,.$$

Now using the addition formula for **Fibonacci** numbers given above, it follows that

$$\begin{aligned} M_{n+k} &= (F_{n-1}F_{k-3} + F_n\ F_{k-2}) + (F_{n-1}F_{k-1} + F_nF_k) + (F_{n-1}F_{k+2} + F_nF_{k+3}) \\ &= (F_{n-1}F_{k-3} + F_{n-1} + \mathrm{F}_{k-1} + F_{n-1}F_{k+2}) + (\mathrm{F}_n\,\mathrm{F}_{k-2} + \mathrm{F}_n\ F_k + F_nF_{k+3}) \\ &= F_{n-1}(F_{k-3} + \mathrm{F}_{k-1} + F_{k+2}) + F_n(\mathrm{F}_{k-2} + F_k +{}_n F_{k+3}) \\ &= F_{n-1}\ M_k + F_n\ M_{k+1}\,. \end{aligned}$$

Hence the theorem is proved.

Theorem 4:

$$M_{2n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$$

Proof: By Theorem 3 we have,

$$\begin{aligned} M_{2n-1} = M_{n+(n-1)} &= F_{n-1}M_{n-1} + F_nM_n \\ &= F_{n-1}M_{n-1} + F_n(L_n + 2F_{n-1}) \\ &= F_{n-1}M_{n-1} + F_nL_n + 2F_nF_{n-1} \\ &= F_{n-1}M_{n-1} + F_{2n} + 2F_nF_{n-1}\,. \end{aligned}$$

Now applying *Theorem 3* to M_{n-1}, we have

$$M_{n-1} = M_{(n-1)+0} = F_{n-2}M_0 + F_{n-1}M_1 = 4F_{n-2} + F_{n-1} \text{ and}$$

$$4F_{n-2} + F_{n-1} = 4(F_n - F_{n-1}) + F_{n-1} = -3F_{n-1} + 4F_n.$$

Hence, $M_{2n-1} = F_{2n} + F_{n-1}(-3F_{n-1} + 4F_n) + 2F_nF_{n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$

Theorem 4. The Subtraction formula for **Mulatu** numbers

$$M_{n-k} = 4F_{n-k+1} - 3F_{n-k}$$

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Proof: $M_{n-k} = M_{(n-k)+0}$ and hence by Theorem 3, we have

$$M_{n-k} = F_{n-k-1}M_0 + F_{n-k}M_1$$

$$= 4F_{n-k-1} + F_{n-k}$$

$$= 4(F_{n-k-1} + F_{n-k}) - 3F_{n-k}$$

$$= 4F_{n-k+1} - 3F_{n-k}.$$

Corollary 1.

$$M_n = 4F_{n+1} - 3F_n.$$

Theorem 5.

$$F_{2n} - M_n \mathrm{F}_{n+1} - F_{n+1}F_n = -L^2{}_n$$

Proof: We use the identities listed above to prove the theorem.

Note that $F_{2n} - M_n \mathrm{F}_{n+1} - F_{n+1}F_n = F_nL_n - M_n\mathrm{F}_{n+1} - F_{n+1}F_n$

$$= F_n(F_{n-1} + F_{n+1}) - F_{n+1}(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) -$$

$$(F_n + F_{n-1})(\mathrm{F}_{n+1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(F_{n-1} + F_n + F_{n-1}) -$$

$$(F_n + F_{n-1})(\mathrm{F}_n + F_{n-1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(2F_{n-1} + F_n) - (F_n + F_{n-1})(\mathrm{F}_n + 4F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= 2F_nF_{n-1} + F^2{}_n -$$

$$F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F_{n-1}{}^2 - F^2{}_n - F_{n-1}F_n$$

$$= -F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1}$$

$$= -(F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1})$$

$$= -(F_n + 2F_{n-1})^2$$

$$= -(F_n + F_{n-1} + F_{n-1})^2$$

$$= -(F_{n+1} + F_{n-1})^2$$

$$= -L^2{}_n$$

The following result deals with the Double -angle type formula. It is rather an amazingly interesting strong result.

Theorem 6. Fundamental identity.

$$M_{2n} = M_nL_n + 4(-1)^{n+1}$$

Proof: By *Theorem 3*, $M_{2n} = M_{n+n} = F_{n-1}M_n + F_nM_{n+1}$. Again applying Theorem 3, to

M_{n+1} and using $L_n = F_{n+1} + F_{n-1}$, we get

$$M_{2n} = F_{n-1}M_n + F_n(F_{n-1}M_1 + F_nM_2)$$

$$= F_{n-1}M_n + F_n(F_{n-1} + 5F_n).$$

$$= F_{n-1}M_n + F_nF_{n-1} + 5F^2{}_n.$$

$$= ((L_n - F_{n+1})M_n + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F_{n+1}M_n + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(L_n + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F^2{}_{n-1} + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1}) + 5F^2{}_n$$

From the proof of *Theorem 5*, we know that $F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1} = L^2{}_n$.

Hence $M_{2n} = L_nM_n - L^2{}_n + 5F^2{}_n$. Now using that $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$ it easily follows that $M_{2n} = L_nM_n + 4(-1)^{n+1}$.

Remark 2: Note that using Corollary 1, we can also express M_{2n} as follows:

$$M_{2n} = 4F_{2n+1} - 3F_{2n}.$$

Corollary 2.

$$M_{2n} = L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$$

Proof: We have

$$M_{2n} = M_n L_n + 4(-1)^{n+1}$$
$$= (L_n + 2F_{n-1})\, L_n + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}L_n + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}(F_{n+1} + F_{n-1}) + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}(F_n + F_{n-1} + F_{n-1}) + 4(-1)^{n+1}$$
$$= L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$$

Corollary 3. Square Expansion

$$M^2{}_n = M_{2n} + 2M_nF_{n-1} + 4(-1)^n$$

Proof: Note that

$$M^2{}_n = M_nM_n = M_n(L_n + 2F_{n-1}) = M_nL_n + 2M_nF_{n-1}.$$

Hence the corollary follows by *Theorem 6*.

Theorem 7.

$$\frac{9F_n^{\ 2} + L^2{}_2 + 4F^2{}_{n-1}}{2} = L_nM_n + 4(-1)^{n+1}$$

Proof:

$$\frac{9F_n^{\ 2} + L^2{}_n + 4F^2{}_{n-1}}{2} = \frac{5F_n^{\ 2} + L^2{}_n + 4F_n^{\ 2} + 4F^2{}_{n-1}}{2}$$
$$= \frac{5F_n^{\ 2} + L_n^{\ 2}}{2} + 2F_n^{\ 2} + F^2{}_{n-1}$$

Now by addition formula for Lucas numbers and Fibonacci numbers given above, we get

$$\frac{9_n^{\ 2} + L^2{}_n + 4F^2{}_{n-1}}{2} = L_{2n} + 2F^2{}_{n-1} + 2F_n^{\ 2} = L_{2n} + 2F_{2n-1}.$$

Now using

$M_n = L_n + 2F_{n-1}$, we obtain that

$$\frac{9_n^{\ 2} + L^2{}_n + 4F^2 n - 1}{2} = M_{2n}$$

Thus theorem follows by Theorem 6.

Some Open Questions.

(1) Are there any more triangular numbers in Mulatu numbers other than 1, 6, 28, and 45? If so, are they finite or infinite?
(2) Are there any more Fermat numbers in Mulatu numbers other than 5 and 17? If so, are they finite or infinite?
(3) Are there any more Fibonacci numbers in Mulatu numbers other than 1 and 5? If so, are they finite or infinite?
(4) Are there any more Lucas numbers in Mulatu numbers other than 1 and 11? If so, are they finite or infinite?
(5) Observe that for n= 1,6,11, 16, and 21 all M, F, and L numbers have the same last digit. Is this pattern finite or infinite?

References:

[1]. Mulatu Lemma, The Mulatu Numbers, *Advances and Applications in Mathematical Sciences,* Volume 10, issue 4, august 2011, page 431-440.
[2]. Burton, D. M., *Elementary number theory*. New York City, New York: McGraw-Hill. 1998.