EPH - International Journal of Mathematics and Statistics
ISSN (Online): 2208-2212
Volume 5 Issue 2 August 2019

DOI:https://doi.org/10.53555/eijms.v5i1.33

NOTE ON THE TWO CELEBRITY NUMBERS, PERFECT NUMBERS AND TRIANGULAR NUMBERS

Mulatu Lemma[1*] and Raymond Quillet[2]
*[*12]Department of Mathematics, Savannah State University Savannah, Ga, 31044, USA*

***Corresponding Author:-**

Abstract:-

Mathematicians have been fascinated for centuries by the properties and patterns of numbers. They have noticed that some numbers are equal to the sum of all of their factors (not including the number itself). Such numbers are called perfect numbers. Thus a positive integer is called a perfect number if it is equal to the sum of its proper positive divisors. The search for perfect numbers began in ancient times. The four perfect numbers 6, 28, 496, and 8128 seem to have been known from ancient times. In this paper, we will investigate some important properties of perfect numbers. We give easy and simple proofs of theorems using finite series. We give our own alternative proof of the well-known Euclid's Theorem (Theorem I). We will also prove some important theorems which play key roles in the mathematical theory of perfect numbers.

Key Words:-*Prime Numbers, Perfect numbers, and Triangular numbers. Perfect square, Pascal Triangles.*

1. INTRODUCTION AND BACKGROUND

Throughout history, there have been studies on perfect numbers. It is not known when perfect numbers were first studied and indeed the first studies may go back to the earliest times when numbers first aroused curiosity. It is rather likely, although not completely certain, that the Egyptians would have come across such numbers naturally given the way their methods of calculation worked. Although, the four perfect numbers 6, 28, 496 and 8128 seem to have been known from ancient times and there is no record of these discoveries. The First recorded mathematical result concerning perfect numbers which is known occurs in Eculid's Elements written around 300BC.

Theorem 1. If 2^k -1 (k>1) is prime, then $n = 2^{k-1}(2^k-1)$ is a perfect number.

Proof: We will show that n = sum of its proper factors.
We will find all the proper factors of $2^{k-1}(2^k-1)$, and add them. Since 2^k-1 is prime, let $p = 2^k-1$. Then $n = p(2^k-1)$
Let us list all factors of 2^{k-1} and other proper factors of n as follows.

Factors of 2*k*-1	Other Proper Factors
1	p
2	$2p$
22	2^2p
23	2^3p
:	:
:	:
2*k*-1	2*k*-2 *p*

Adding the first column, we get:

$$1+2+2^2+2^3\ldots+2^{k-3}+2^{k-2}+2^{k-1}$$
$$=2^k-1$$
$$=p$$

Adding the second column, we get:

$$p+2p+2^2p+2^3p\ldots+2^{k-4}p+2^{k-3}p+2^{k-2}p$$
$$=p\left(1+2+2^2+\ldots+2^{k-2}\right)$$
$$=\left(2^{k-1}-1\right)p$$

Now adding the two columns together, we get:

$$p+p\left(2^{k-1}-1\right)$$
$$=p\left(1+2^{k-1}-1\right)$$
$$=p\left(2^{k-1}\right)$$
$$=n$$

Hence. n is a perfect number.

Remark I: A question can be raised if k is prime by itself
$\Rightarrow 2^{k-1}(2^k-1)$ is a perfect number. The answer is negative as it will be easily shown that it does not work for k=11.

Corollary 1:: If 2^k-1 is prime, then $n=2^{k-1}+2^k+2^{k+1}\ldots+2^{2k-2}$ is a perfect number.
Proof: We have:

$$n=2^{k-1}+2^k+2^{k+1}\ldots+2^{2k-2}=2^{k-1}\left(1+2+2^2+2^3\ldots+2^{k-1}\right)$$
$$n=2^{k-1}\left(2^k-1\right)$$

$\Rightarrow$ *n* is a perfect number by Theorem 1.

Remark II: Every even perfect number *n* is of the form $n = 2^{k-1}(2^k-1)$. We will not prove this, but we will accept and use it. So, the converse to Theorem 1 is also true. This is called Euler's Theorem.
Next we will show how **Remark II** applies to the first four perfect numbers. Note that:

$$6=2\cdot3=2^1\left(2^2-1\right)=2^{2-1}\left(2^2-1\right)$$
$$28=4\cdot7=2^2\left(2^3-1\right)=2^{3-1}\left(2^3-1\right)$$
$$496=16\cdot31=2^4\left(2^5-1\right)=2^{5-1}\left(2^5-1\right)$$
$$8128=64\cdot127=2^6\left(2^7-1\right)=2^{7-1}\left(2^7-1\right)$$

Theorem 2: The sum of the reciprocals of the factors of a perfect number is n is equal to 2.
Proof: Let $n = 2^{k-1}(2^k-1)$ where $p = 2^k-1$ and is prime. Let us list all the possible factors of n.

Factors of 2^{k-1}	Other Factors
1	p
2	p
22	2^2p
:	:
:	:
2*k*-1	2*k*-1 p

Sum of reciprocals of factors 2^{k-1}

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}}{2\left(2^{k-1}\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}\right)}\ldots+\frac{1}{\left(2^{k-1}\right)}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-2}}{2^{k-1}}+\frac{2^{k-3}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}}$$
$$=\frac{2^k-1}{2^{k-1}}=\frac{p}{2^{k-1}}$$

Sum of reciprocals of other factors

$$\frac{1}{p}+\frac{1}{2p}+\frac{1}{2^2p}+\frac{1}{2^3p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}}{2\left(2^{k-1}p\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}p\right)}\ldots+\frac{1}{\left(2^{k-1}p\right)}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-2}}{2^{k-1}p}+\frac{2^{k-3}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}p}$$
$$=\frac{2^k-1}{2^{k-1}p}=\frac{p}{2^{k-1}p}=\frac{1}{2^{k-1}}$$

Now the sums of reciprocals of all factors are equal to:

$$=\frac{p}{2^{k-1}}+\frac{1}{2^{k-1}}$$
$$=\frac{p+1}{2^{k-1}}$$
$$=\frac{2^k-1+1}{2^{k-1}}$$
$$=\frac{2^k}{2^{k-1}}=2$$

Corollary 2. No power of a prime can be a perfect number.
Proof: Let p be prime and let $n = p^k$. The factors of n are $1, p, p^2, p^3 \ldots p^k$.
Now, we have:

$$\begin{aligned}
& 1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\ldots+\frac{1}{p^k} \\
&= 1+\frac{p^{k-1}+p^{k-2}+p^{k-3}\ldots+p+1}{p^k} \\
&= 1+\frac{p^k-1}{p^k(p-1)} \\
&\le 1+\frac{p^k-1}{p^k} \\
&= 1+\frac{p^k}{p^k}-\frac{1}{p^k} \\
&= 1+1-\frac{1}{p^k} \\
&= 2-\frac{1}{p^k} < 2.
\end{aligned}$$

Therefore, n is not a perfect number.

Theorem 3: If n is a perfect number such that $n = 2^{k-1}(2^k-1)$, then the product of the positive divisor's of n is equal ton^k.
Proof: We list factors of n as in Theorem 2

Factors of 2^{k-1}	Other Factors
1	P
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
$2k$-1	$2k$-1 p

Product of column 1 =

$$1*2*2^2*2^3\ldots*2^{k-1} = 2^{1+2+3\ldots+(k-1)} = 2^{\frac{k(k-1)}{2}}$$

Product of column 2 =

$$\begin{aligned}
& p\cdot 2p\cdot 2^2p\ldots\cdot 2^{k-1}p \\
&= p^k(1\cdot 2\cdot 2^2\ldots 2^{k-1}) \\
&= p^k(2^{\frac{k(k-1)}{2}}),
\end{aligned}$$

Therefore the products of both columns are

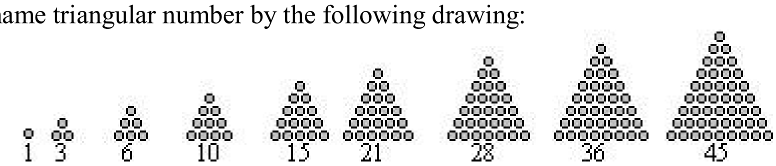
$$\begin{aligned}
&= 2^{\frac{k(k-1)}{2}}\cdot p^k\cdot 2^{\frac{k(k-1)}{2}} \\
&= 2^{k(k-1)}\cdot p^k \\
&= (2^{k-1}\cdot p)^k \\
&= n^k.
\end{aligned}$$

Triangular Numbers
The triangular numbers are formed by partial sum of the series 1+2+3+4+5+6+7…. +n. In other words, triangular numbers are those counting numbers that can be written as T_n =
1+2+3+…+ n. So,
T_1= 1
T_2= 1+2=3
T_3= 1+2+3=6

$T_4 = 1+2+3+4=10$
$T_5 = 1+2+3+4+5=15$
$T_6 = 1+2+3+4+5+6= 21$
$T_7 = 1+2+3+4+5+6+7= 28$
$T_8 = 1+2+3+4+5+6+7+8= 36$
$T_9 = 1+2+3+4+5+6+7+8+9=45$
$T_{10} = 1+2+3+4+5+6+7+8+9+10=55$
.
These are the first 100 triangular numbers:

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210
231	253	276	300	325	351	378	406	435	465
496	528	561	595	630	666	703	741	780	820
861	903	946	990	1035	1081	1128	1176	1225	1275
1326	1378	1431	1485	1540	1596	1653	1711	1770	1830
1891	1953	2016	2080	2145	2211	2278	2346	2415	2485
2556	2628	2701	2775	2850	2926	3003	3081	3160	3240
3321	3403	3486	3570	3655	3741	3828	3916	4005	4095
4186	4278	4371	4465	4560	4656	4753	4851	4950	5050

You can illustrate the name triangular number by the following drawing:



Special Triangular numbers

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210
231	253	276	300	325	351	378	406	435	465
496	528	561	595	630	666	703	741	780	820
861	903	946	990	1035	1081	1128	1176	1225	1275
1326	1378	1431	1485	1540	1596	1653	1711	1770	1830
1891	1953	2016	2080	2145	2211	2278	2346	2415	2485
2556	2628	2701	2775	2850	2926	3003	3081	3160	3240
3321	3403	3486	3570	3655	3741	3828	3916	4005	4095
4186	4278	4371	4465	4560	4656	4753	4851	4950	5050

You see:
The even triangular numbers in red and the odd numbers in black form pairs in the usual sequence.

Theorem 4. Every triangular number is a **binomial coefficient .**
Proof without words: - Refer to the following Pascal's Triangle [and see the red colored numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Theorem 5. If T_m and T_n are triangular numbers, then

$$T_{m+n} = T_m + T_n + mn$$

For m and n positive integers.

Proof:

Note: $T_m = \frac{m(m+1)}{2}$ & $T_n = \frac{n(n+1)}{2}$. Then

$$\begin{aligned} T_m + T_n + mn &= \frac{m(m+1)}{2} + \frac{n(n+1)}{2} + mn \\ &= \frac{m^2+m+n^2+n}{2} + mn \\ &= \frac{m^2+m+n^2+n+2mn}{2} = \frac{m^2+2mn+n^2+m+n}{2} \\ &= \frac{(m+n)(m+n)+(m+n)}{2} = \frac{(m+n)[m+n+1]}{2} = T_{m+n} \end{aligned}$$

Theorem 6: If T_m and T_n are triangular numbers, then

$$T_{mn} = T_m T_n + T_{m-1} T_{n-1}$$

Proof:

Note: $T_m = \frac{m(m+1)}{2}$ and $T_n = \frac{n(n+1)}{2}$. Then

$$T_m T_n + T_{m-1} T_{n-1} = \frac{m(m+1)}{2}\frac{n(n+1)}{2} + \frac{(m-1)m}{2}\frac{(n-1)n}{2}$$

$$= \left(\frac{m^2+m}{2}\right)\left(\frac{n^2+n}{2}\right) + \left(\frac{m^2-m}{2}\right)\left(\frac{n^2-n}{2}\right)$$

$$= \left[\frac{m^2n^2+mn^2+nm^2+mn}{4}\right] + \left[\frac{m^2n^2-mn^2-nm^2+mn}{4}\right]$$

$$= \frac{2m^2n^2+2mn}{4} = \frac{2mn(mn+1)}{4} = \frac{mn(mn+1)}{2}$$

$$= T_{mn}$$

Theorem 7. Every even perfect number *n is* a triangular number.

Proof: n is a perfect number $\Rightarrow$ n= $2^{k-1}(2^k-1)$ by Remark III. Hence, n= $\frac{2^k(2^k-1)}{2} = \frac{(m+1)m}{2}$ where m=2^k-1. Thus n is a triangular number.

Corollary 3. If T is a perfect number, then 8T +1 is a perfect square**.**
Proof: T is a perfect number $\Rightarrow$T is a triangular number.
$\Rightarrow$ T=$\frac{(m+1)m}{2}$ For some positive integer m.

$$\Rightarrow 8T+1 = 4m(m+1)+1$$
$$= 4m^2+4m+1$$
$$=(2m+1)^2$$

Theorem 8. If T_n be triangular numbers for $n \geq 1$, then we have

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$$

Proof:

$$\sum_{n=1}^{\infty} \frac{1}{T_n}$$
$$= \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$
$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 2(1) = 2$$

References

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[2]. Burton, D. M. (1998). *Elementary number theory*. New York City, New