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ON A CLASS OF WEAKLY BERWALD SPECIAL (*A, B*)-METRICS OF SCALAR FLAG CURVATURE

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Abstract:-

In this paper, we study the Finsler space with special (α, β)-metric $F = \alpha + \frac{\beta^2}{\alpha}$ *is scalar flag curvature and we proved that, if it is weakly Berwald if and only if it is Berwald and vanishes flag curvature. Further, we found that this metric is locally Minkowskian.*

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Key Words:-*Finsler space, S-curvature, (α, β)-metrics, Mean Berwald curvature, Homogenous Finsler space.*

1. INTRODUCTION

In Finsler geometry, the flag curvature is an analogue of the sectional curvature in Riemannian geometry. One of the fundamental problem in Riemanian-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature. It is known that every Berwald metric is a Landsberg metric and also, for any Berwald metric the *S*-curvature vanishes. In 2003 X.Chang, X.Mo and Shen have obtained the results on the flag curvature of Finsler metircs of scalar curvature. Then, in 2004, Yoshikawa,Okubo and M.Matsumoto showed the conditions for some (α,β)-metrics to be weakly Berwald. Recently Narasimhamurthy.S.K[8] has studied the class of weakly Berwld (α,β)-metrics of scalar flag curvature.

The main purpose of the present paper is to charecterize the weakly Berwald special (α,β)metrics are of scalar flag curvature. Then, we proved that it vanishes scalar flag curvature and Berwald, then it is locally Minkowskian.

2. Preliminaries:

This section includes the basics of Finsler spaces and the concepts of *S*-curvatre.

Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α,β)-metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form.

Now, let $\varphi(s)$ be a positive c^∞ function on $(-b_0,b_0)$. For a number $b \in [0,b_0)$, let

$$\phi = -(Q - sQ')n\Delta + 1 + sQ - (b^2 - s^2)(1 + sQ)Q'', \tag{2.1}$$

where, $\Delta = 1 + sQ + (b^2 - s^2)Q`$ and $Q = \frac{Q}{\phi - s\phi'}$.

Let G^i and $\bar{G}^i$ denotes the spray co-efficients of F and α respectively. Then geodesic co-efficients G^i are related to $\bar{G}^i$ of by

$$G^i = \tilde{G}^i + \alpha Q s^i_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\}, \tag{2.2}$$

where $\bar{G}^i$ denote the spray coefficients of α.

We shall denote $r_{00} = r_{ij}y^iy^j$, $s_{i0} = s_0 = s_iy^i$ and $\Theta = \frac{Q - sQ'}{2\Delta}$.

Also,

$$h_i = \alpha b_i - sy_i, \quad \psi_i = \sqrt{b^2 - s^2}\Delta^{1/2}\left[\frac{\sqrt{b^2 - s^2}}{\Delta^{3/2}}\Phi\right]'$$

$$\psi_2 = 2(n+1)(Q - sQ') + 3\frac{\phi}{\Delta}.,$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of (α,β)-metrics as follows[5].:

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i). \tag{2.3}$$

According to Diecke's theorem[10], a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes, $I = 0$. Clearly, an (α,β)-metric $F = \alpha\varphi(\beta/\alpha)$, $s = \beta/\alpha$ is Riemannian if and only if $\Phi = 0$. (see [21]).

For a Finsler metric $F = F(x,y)$ on a manifold M, the Riemann curvature $R_y = R_k^i \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Let $Rjk = gjiRki$, then $Rjkyj = 0$, $Rjk = Rkj$.

For a flag $\{P,y\}$, where $P = span\{y,u\} \subset T_xM$, the flag curvature $K = K(P,y)$ of F is defined by

$$K(P,y) = \frac{R_{jk}(x,y)u^ju^k}{F^2(x,y)h_{jk}(x,y)u^ju^k}$$

$$h_{jk} = g_{jk} - F^{-2}g_{jp}y^pg_{kq}y^q., \tag{2.4}$$

where,

We say that Finsler metric F is of scalar flag curvature if the flag curvature $K = K(x,y)$ is independent of the flag P. By the definition, F is of scalar flag curvature $K = K(x,y)$ if and only if in a standard local coordinate system,

$$R_k^i = KF^2h^i_k, \tag{2.5}$$

where, $h^i_k = g^{ij}h_{jk} = g^{ij}FF_{y^j y^k}$([4],[5]).

The Schur lemma[4] in Finsler geometry tell us that, in dimension $n \geq 3$, if F is of isortopic flag curvature, $K = K(x)$, then it is of constant flag curvature, $K = constant$. The Berwald curvature $B_y = B_{jkl}{}^i\, dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ and mean Berwald curvature $E_y = E_{ij}dx^i \otimes dx^j$ are defined respectively by

$$B^i_{jkl} = \frac{\partial^3 G^i(x,y)}{\partial y^j \partial y^k \partial y^l}, \quad E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right) = \frac{1}{2}B^m_{ijm}.$$

A Finsler metric F is called weak Berwald metric if the mean Berwald curvature vanishes, i,e., $(E = 0)B = 0$. A Finsler metric F is said to be of isotropic mean Berwald curvature if $F = \frac{1}{2}(n+1)c(x)F^{-1}h$, where $c = c(x)$ is a scalar function on the manifold M.

Theorem 2.1. [8],[9] *For special* (α,β)*-metric* $F = \alpha + k\frac{\beta^2}{\alpha}$, *where* k 6= 0, *is constant and Matsumoto metric (*2*nd appx)* $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ *on n-dimensional manifold M. Then the following are equivalent:*
(a) F is of isotropic S-curvature, $S = (n+1)c(x)F$;
(b) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}c(x)F^{-1}h$;
(c) β *is a killing* 1*-form with constant length with respect to* α, *i,e.,* $r_{00} = 0$ *and* $s_{00} = 0$;
(d) S-curvature vanishes, $S = 0$;
(e) F is a weak Berwald metric $E = 0$; *where,* $c = c(x)$ *is scalar function on the manifold M.*

Note that, the discussion in [3], [19] doesn't involve whether or not F is Berwald metric. By the definition[7], Berwald metrics must be weak Berwald metrics but the converse is not true. For this observation, we further study the Finsler metrics $F = \alpha + \frac{\beta^2}{\alpha}$.

For a Finsler metric $F = F(x,y)$ on and n-dimensional manifold M, the Busemann-Hausdroff volume form $dV_F = \sigma_F(x)dx^1 \wedge \ldots\ldots \wedge dx^n$ is given by

$$\sigma_F(x) = \frac{vol(B^n(1))}{vol\{(y^i)\in R^n \mid F(x,y)<1\}},$$

Vol denotes the Euclidean volume in R^n. The S-curvature is given by

$$S(x,y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{ly\partial\,(ln\sigma_F)}{\partial x^m}. \tag{2.6}$$

Clearly, the mean Berwald curvature $E_y = E_{ij}dx^i \otimes dx^j$ can be charecterized by use of S-curvature as follows:

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j},$$

A Finsler metric F is said to be isotropic S-curvature if $S = (n+1)c(x)F(x,y)$, where $c = c(x)$ is a scalar function on the manifold M. S-curvature is closely related to the flag curvature. We use the following important results proved by [7].

Theorem 2.2. *Let* (M,F) *be an n-dimensional Finsler manifold of scalar flag curvature with flag curvature* $K = K(x,y)$. *Suppose that the S-curvature is isotropic,* $S = (n+1)c(x)F(x,y)$, *where* $c = c(x)$ *is a scalar function on M. Then there is a scalar function* $\sigma(x)$ *on M such that*

$$K = \frac{3_{c_{x^m}}(x)y^m}{F(x,y)} + \sigma(x). \tag{2.7}$$

This shows that S-curvature has important influence on the geometric structures of Finsler metrics. For a Finsler metric F, the Landsberg curvature $L = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ and the mean Landsberg curvature $L = J_k dx^k$ are defined respectively by

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}, \quad J_k = g^{ij}L_{ijk},$$

A Finsler metric F is called weak Landsberg curvature metric if the mean Landsberg curvature vanishes, i,e., $(J = 0)L = 0$. For an (α,β)-metric $F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$, Li Benling and Z,Shen[7]

obtained the following formula of the mean Landsberg curvature

$$J_i = -\frac{1}{2\Delta\alpha^4}\{\frac{2\alpha^2}{b^2-s^2}\left[\frac{\phi}{\Delta} + (n+1)(Q - sQ')\right](s_0 + r_0)h_i + \frac{\alpha}{b^2-s^2}\left[\psi_1 + s\frac{\phi}{\Delta}\right](r_{00} - 2Q\alpha s_0)h_i$$
$$\alpha\left[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0)\right] + \alpha^2\Delta s_{i0} + \left[\alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\right]\frac{\phi}{\Delta}\}. \tag{2.8}$$

Besides, they also obtained

$$\bar{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}\{\psi_1(r_{00} - 2\alpha Q s_0) + \alpha\psi_2(r_0 + s_0)\}. \tag{2.9}$$

The horizontal covarient derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α respectively are given by

$$J_{i;m} = \frac{\partial J_i}{\partial x^m} - J_l \Gamma^l_{im} - \frac{\partial J_i}{\partial y^l} N^l_m, \quad J_{i|m} = \frac{\partial J_i}{\partial x^m} - J_l \bar{\Gamma}^l_{im} - \frac{\partial J_i}{\partial y^l} \bar{N}^l_m,$$

where,

$$\Gamma^l_{ij} = \frac{\partial G^l}{\partial y^i \partial y^j}, \quad N^l_j = \frac{\partial G^l}{\partial y^j} \quad and \quad \bar{\Gamma}^l_{ij} = \frac{\partial \bar{G}^i}{\partial y^i \partial y^j}, \quad \bar{N}^l_j = \frac{\partial (G)^l}{\partial y^j}.$$

Further we have,

$$J_{i;m} y^m = \{J_{i|m} - J_l(\Gamma^l_{im} - \bar{\Gamma}^l_{im}) - \frac{\partial J_i}{\partial y^l}(N^l_m - \bar{N}^l_m)\} y^m$$
$$= J_{i|m} y^m - J_l(N^l_i - \bar{N}^l_i) - 2\frac{\partial J_i}{\partial y^l}(G^l - \bar{G}^l).$$

If a Finsler metric F is of constant flag curvature K(see[5]), then

Ji;mym + KF2Ii = 0.

So, if can (α,β)-metric $F = \alpha\phi(s),\ s = \frac{\beta}{\alpha}$, is of constant flag curvature K, then

$$J_{i|m} y^m - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} - 2\frac{\partial J_i}{\partial y^i}(G^l - \bar{G}^l) + K\alpha^2\phi^2 I_i = 0.$$

Contracting the above equation by b^i yields the following equation

$$\bar{J}_{i|m} y^m - J_i a^{ik} b_{k|m} y^m - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} b^i - 2\frac{\partial \bar{J}}{\partial y^l}(G^l - \bar{G}^l) + K\alpha^2\phi^2 I_i b^i = 0 \quad (2.10)$$

3. Characterization of Weakly Berwald special (α,β)-metrics of scalar flag curvature:

In this section, for a n-dimensional Finsler manifold ($n \geq 3$), we characterize the class of weakly Berwald special (α,β)-metric of scalar flag curvature. So first we prove the following lemma;

Lemma 3.1. *Let $F = \alpha + \frac{\beta^2}{\alpha}$ be a Finsler space with (α,β)-metric, then it is non-Randers type, if* Θ 6= 0.

Proof: Consider the Finsler metric $F = \alpha + \frac{\beta^2}{\alpha}$. By direct computation using (2.1) we have

$$\Theta = \frac{-A\phi}{(1-s^2)^4},$$

where,

$$\phi = 1 + s^2$$

$$A = (3n+1)s^5 - 16s^4 - \left(4(n+1) + 8nb^2 + 4b^2\right)s^3 + 4(4b^2-1)s^2 + 4b^2.$$

Assume that $\Theta = 0$, then $A = 0$ obviously and so multiplying $A = 0$ with α^5, which yields.

$4\beta^5(1 + 12n) - [(n + 1) + 2nb^2 + B^2]\beta^3\alpha^2 + \alpha[4b^2\alpha^4 + 4(4b^2 - 1)\beta^2\alpha^2 - 16\beta^4] = 0$. Hence we obtain,

$$\beta^5(1+12n) - \left((n+1) + (2n+1)b^2\right)\beta^3\alpha^2 = 0, \quad (3.1)$$

and

$$b^2\alpha^4 + (4b^2 - 1)\beta^2\alpha^2 - 4\beta^4 = 0. \quad (3.2)$$

Clearly, observe that in equation (3.1) β^5 is divisible by α^2, which is contraction to $\Theta = 0$.
Therefore, F must be a non-Randers type.
The main idea of this lemma(3.1), now we can prove the following theorem.

Theorem 3.3. *Let $F = \alpha + \frac{\beta^2}{\alpha}$ be a Finsler space with (α,β)-metric is of scalar flag curvature $K = K(x,y)$, then F is weakly Berwald metric if and only if F is Berwald metric and $K = 0$.*
In this case, F must be locally Minkowskian.

Proof. From lemma(3.1) and theorem(2.2), we know that Finsler space with (α,β)-metrics $F = \alpha + \frac{\beta^2}{\alpha}$ cannot represents the Riemannian mertric if β 6= 0.
Then the proof of this theorem sufficient part is trival. and so we only prove the necessary condition.
Assume that the metric F is weakly Berwald. By lemma(3.1) and the isotropic S-curvature, $S = (n + 1)c(x)F$ with $c(x)$ is constant and

$$r_{00} = 0, \qquad s_0 = 0. \quad (3.3)$$

Again theorem(2.2) and the equation (2.7) F must be of isotropic flag curvature $K = \sigma(x)$.
Further, eqution(2.10),we can simply using (2.2),(2.8) and (2.9) as follows.

$$G^i - \bar{G}^i = \alpha Q s^i_0, \quad J_i = \frac{-\Theta s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0.$$

In addition from(2.3) we obtain

$$I_i b^i = -\frac{\Theta(\phi - s\phi')}{2\Delta F}(b^2 - s^2), \qquad (3.4)$$

thus, equation(2.10) can be expressed as follows:

$$\frac{\Theta s_{i0}}{2\Delta\alpha} a^{ik} s_{k0} + \frac{\Theta s_{l0}}{2\Delta\alpha}\left(sQs_0^l + Q' s_0^l (b^2 - s^2)\right) - KF\frac{\Theta}{2\Delta}(\phi - s\phi')(b^2 - s^2) = 0.$$

Again by using lemma(3.1) we get

$$s_{i0}s_0^i + s_{l0}(\alpha Q s_0^l) b_i - KF\alpha(\phi - s\phi')(b^2 - s^2) = 0.$$

Here, notice that; $F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$. Thus we have,

$$s_{i0}s_0^i \Delta - K\alpha^2 \phi(\phi - s\phi')(b^2 - s^2) = 0. \qquad (3.5)$$

For Finsler space with (α,β)-metric $F = \alpha + \frac{\beta^2}{\alpha}$ we have,

$$\Delta = \frac{\phi(1+2b^2-3s^2)}{(1-s^2)^2},$$

Then equation (3.5) becomes,

$$(1 + 2b^2 - 3s^2)s_{i0}s_0^i - K\alpha^2(b^2 - s^2)(1 - s^2)^3 = 0.$$

On multiplying above equation with α^6, which yields.

$$-b^2\alpha^8 + \{\beta^2(1+3b^2) + (1+2b^2)s_{i0}s_0^i\}\alpha^6 - 3\beta^2\{(1+2b^2) + s_{i0}s_0^i\}\alpha^4 + K\beta^6(3+b^2)\alpha^2 = K\beta^8. \quad (3.6)$$

Here, we observe that,the leftside of equation (3.6) is divisible by α^2. Hence we obtain that the flag curvature $K = 0$, because β^8 is not divisible by α^2. Substituting $K = 0$ into equation (3.5) we have $s_{i0}s_0^i = a_{ij}(x)s_0^j s_0^i$= 0. Because $(a_{ij}(x))$ is positive defnite, so s_0^i= 0, i,e., β is closed. By equation (3.3), we know that β is parallel with respect to α. Then F is Berwald mertic if β 6= 0. Hence, Fmust be locally Minkowskian.

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