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A NEW DIFFERENTIAL OPERATOR FOR SOLVING MULTI SINGULAR INITIAL VALUE PROBLEMS OF SECOND ORDER

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Abstract:-

Modified Adomian Decomposition Method (MADM) was used in this research to solve singular initial value problems in the second-order ordinary differential equation. In the beginning, we studied the genral equation, and we gave several illustrations for this equation. In addition, we studied some different cases of second order ordinary differential equations which we got from genral equation. From these cases, we got different types of equations such as multi singular initial value problems, like Bessel's equation from half order and other equations as well. Using the examples under investigation, we conclude that the solution is converging towards the exact solution.

Key words:- “Multi singular initial value problems, Modified Adomian Decomposition Method second order nonlinear ordinary differential equation.”

I. INTRODUCTION

Singular initial value problems for second order ordinary differential equations arise from nonlinear phenomena in physics and mechanics (see [13] and the references therein). Singular problems studies are regarded as new as they started in the middle of 1970s.

For investigating the numerical singular differential equations, researchers have proposed many various methods and techniques. I.T Abu-Zaid, and M.A. El-Gebeily [8] had earlier solved singular two point boundary value problem using finite difference approximation, and L.U. Junfeng [11] had solved singular differential equation using variational iteration method. In addition, there are other methods that have given a general study to construct the exact solution and series solution of singular two point boundary value problems. For example, cubic splines by A.S.V. Ravi Kanth and Y.N. Reddy [3], Sine-Galerkin method and Homotopy perturbation method by K. Al-Khaled [10]. The decomposition method introduced by George Adomian at the beginning of 1980s has received immense attention in the past two decades. G. Adomian [4,5,6] asserts that the decomposition method provides an efficient and convenient method for generating approximate series solution to a wide class of differential equations which converges. Singular problem got the interest of researches to apply this method. For example, in [1,2] A.M. Wazwaz presents several researches on singular problem via using (MADM) with differential operators. In [14] Y.Q. Hasan et.al. solved second ordinary differential equation by (MADM). Also in [15] Y.Q. Hasan and L.M. Zhu solved singular boundary value problems of higher-order by modified Adomian decomposition method. A modification on (ADM) was introduced by [12]. Another modification applied on the method was proposed in [9] to solve singular initial value Emden-Fowler type equations of second order. The aim of this research is to find the solution of multi singular initial value problem and many kinds of the second order ordinary differential equation by using a new reliable modification of Adomian decomposition method(ADM). For this reason, a new differential operator is proposed which can be used for solving such equation.

II. MODIFIED ADOMAINDECOMPOSITION METHOD

Assume the singular initial value problem of second order ordinary differential equation,

$$y'' + \frac{n+m}{x}y' + \left(k + \frac{m(n-1)}{x^2} + \frac{n-m}{x}\sqrt{k}\tan\sqrt{kx}\right)y = g(x, y), \quad (1)$$

$$y(0) = a_0, y'(0) = a_1,$$

where $g(x,y)$ is a real function and a_0, a_1 are given constants and $m \geq 0, n \geq 1$.

We suggest the new differential operator as below

$$L(.) = \frac{1}{x^n \cos\sqrt{kx}} \frac{d}{dx} x^{n-m} \cos^2\sqrt{kx} \frac{d}{dx} \frac{x^m}{\cos\sqrt{kx}} (.), \quad (2)$$

we can write eq.(1) in the form

$$Ly = g(x,y). \quad (3)$$

The inverse operator L^{-1} is therefore a two-fold integral operator, as below

$$L^{-1}(.) = \frac{\cos\sqrt{kx}}{x^m} \int_0^x x^{m-n} \cos^{-2}\sqrt{kx} \int_0^x x^n \cos\sqrt{kx} (.) dx dx. \quad (4)$$

By applying L^{-1} on (3), we have

$$y(x) = \gamma(x) + L^{-1}(g(x,y)), \quad (5)$$

such that

$$L(y(x)) = 0.$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $g(x,y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (6)$$

and

$$g(x, y) = \sum_{n=0}^{\infty} A_n, \quad (7)$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen [7] to formulate Adomian polynomials. The following algorithm:

$$\begin{aligned} A_0 &= G(y_0), \\ A_1 &= y_1 G'(y_0), \\ A_2 &= y_2 G'(y_0) + \frac{1}{2!} y_1^2 G''(y_0), \\ A_3 &= y_3 G'(y_0) + y_1 y_2 G''(y_0) + \frac{1}{3!} y_1^3 G'''(y_0) \\ &\dots \end{aligned}, \quad (8)$$

Can be used to construct Adomian polynomials, when $G(y)$ is a nonlinear function. By substituting (6) and (7) into (5) gives,

$$\sum_{n=0}^{\infty} y_n(x) = \gamma(x) + L^{-1} \sum_{n=0}^{\infty} A_n \tag{9}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = \gamma(x),$$

$$y_{(n+1)} = L^{-1}A_n, n \geq 0, \tag{10}$$

which gives $y_0 = \gamma(x), y_1 = L^{-1}A_0, y_2 = L^{-1}A_1,$

$$y_3 = L^{-1}A_2, \tag{11}$$

From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (7) can be immediately obtained. For numerical purposes, the n-term approximate

$$\psi_n = \sum_{n=0}^{n-1} y_n, \tag{12}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear differential equations with initial value problem.

III. DISCUSSION OF MADM AND ITS APPLICATIONS

In order to achieve the aim of this research, we have firstly studied the eq.(1), giving illustrative examples. Then we studied its partial cases as shown below.

First: we will give an example on the eq.(1)

Problem 1. When $k = 2, n = 3, m = 4$, in eq.(1) we get

$$y'' + \frac{7}{x}y' + (2 + \frac{8}{x^2} - \frac{\sqrt{2}}{x} \tan \sqrt{2}x)y = (18 + 4x^2 + \frac{8}{x^2} - \frac{\sqrt{2}}{x} \tan \sqrt{2}x)e^{x^2} + x^2 - \ln y.$$

$$y(0) = 1, y'(0) = 0, \tag{13}$$

and $y(x) = e^{x^2}$, is the exact solution.

Eq.(13) can be written as

$$Ly = (18 + 4x^2 + \frac{8}{x^2} - \frac{\sqrt{2}}{x} \tan \sqrt{2}x)e^{x^2} + x^2 - \ln y, \tag{14}$$

where

$$L(.) = \frac{1}{x^3 \cos \sqrt{2}x} \frac{d}{dx} x^{-1} \cos^2 \sqrt{2}x \frac{d}{dx} \frac{x^4}{\cos \sqrt{2}x} (.),$$

and

$$L^{-1}(.) = \frac{\cos \sqrt{2}x}{x^4} \int_0^x x \cos^{-2} \sqrt{2}x \int_0^x x^3 \cos \sqrt{2}x (.) dx dx.$$

Applying L^{-1} on both side of (14) and using initial conditions yields

$$L^{-1}(Ly) = L^{-1}((18 + 4x^2 + \frac{8}{x^2} - \frac{\sqrt{2}}{x} \tan \sqrt{2}x)e^{x^2} + x^2) - L^{-1}(\ln y),$$

$$y(x) = 1 + x^2 + \frac{25x^4}{48} + \frac{x^6}{6} - \frac{18427x^8}{10080} + \frac{77353x^{10}}{226800} + \dots - L^{-1}(\ln y). \tag{15}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (15) gives

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x^2 + \frac{25x^4}{48} + \frac{x^6}{6} - \frac{18427x^8}{10080} + \frac{77353x^{10}}{226800} + \dots - L^{-1}(\ln y), \tag{16}$$

$$y_0 = 1 + x^2 + \frac{25x^4}{48} + \frac{x^6}{6} - \frac{18427x^8}{10080} + \frac{77353x^{10}}{226800} + \dots,$$

$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{17}$$

we get the series of $\ln y$, by Adomian polynomials

$$A_0 = \ln y_0,$$

$$A_1 = \frac{y_1}{y_0},$$

$$A_2 = \frac{y_2}{y_0} - \frac{y_1^2}{2y_0^2}$$

$$\dots, \tag{18}$$

Then

$$\begin{aligned}
 y_0 &= 1 + x^2 + \frac{25x^4}{48} + \frac{x^6}{6} - \frac{18427x^8}{10080} + \frac{77353x^{10}}{226800} + \dots, \\
 y_1 &= \frac{-x^4}{48} - \frac{x^6}{3840} + \frac{203x^8}{11520} - \frac{227x^{10}}{69120} + \dots, \\
 y_2 &= \frac{-x^6}{3840} + \frac{x^8}{3840} - \frac{x^{10}}{23040} + \dots, \\
 &\dots
 \end{aligned}$$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 = 1 + x^2 + \frac{x^4}{2} + \frac{319x^6}{1920} - \frac{24329x^8}{13440} + \frac{1225573x^{10}}{3628800} + \dots$$

When we found y_0, y_1, y_2 , and collected them we noticed that, the solution by (MADM) approximated to exact solution. So, if we continue until y_n , we will get the exact solution.

Note that, the series of exact solution $y(x) = e^{x^2}$, by Taylor series is as below,

$$y(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \dots$$

Second: In this part, we will discuss the cases that we got from eq.(1) and we will give explanation examples.

Case 1. When $k = 0$, in eq.(1) we get multi singular initial value problems

$$\begin{aligned}
 y'' + \frac{m+n}{x}y' + \frac{m(n-1)}{x^2} &= g(x, y), \tag{19} \\
 y(0) = a_0, y'(0) &= a_1,
 \end{aligned}$$

in an operator form eq.(19) become

$$Ly = g(x, y), \tag{20}$$

where

$$L(.) = x^{-n} \frac{d}{dx} x^{n-m} \frac{d}{dx} x^m (.), \tag{21}$$

and

$$L^{-1}(.) = x^{-m} \int_0^x x^{m-n} \int_0^x x^n (.) dx dx. \tag{22}$$

using the inverse operator L^{-1} to both side of eq.(20), we have

$$y(x) = \gamma(x) + L^{-1}(g(x, y)). \tag{23}$$

We will introduce an illustration for the above case and from the examples, it turns out that the method is useful. Moreover, we shall provide a graph as well as a table values that explains the converged solution.

Problem 1. Assume the nonlinear multi singular initial value problem:

$$y'' + \frac{7}{x}y' + \frac{9}{x^2}y = 36x + e^{x^3} - e^y, \tag{24}$$

where

$$L(.) = x^{-4} \frac{d}{dx} x \frac{d}{dx} x^3,$$

so, L^{-1} is given by

$$L^{-1}(.) = x^{-3} \int_0^x x^{-1} \int_0^x x^4 (.) dx dx.$$

Taking L^{-1} to both side of (25) and using the initial conditions we obtain

$$y(x) = \frac{x^2}{25} + x^3 + \frac{x^5}{64} + \frac{x^8}{242} + \dots - L^{-1}e^y. \tag{26}$$

Replace the decomposition series $\sum_{n=0}^{\infty} y_n(x)$ into (26) gives

$$\sum_{n=0}^{\infty} y_n(x) = \frac{x^2}{25} + x^3 + \frac{x^5}{64} + \frac{x^8}{242} + \dots - L^{-1}e^y, \tag{27}$$

the MADM introduce the recursive relation

$$y_0 = \frac{x^2}{25} + x^3 + \frac{x^5}{64} + \frac{x^8}{242} + \dots,$$

$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{28}$$

the nonlinear term e^y , we can get it as below

$$A_0 = ey_0,$$

$$A_1 = y_1ey_0,$$

$$A_2 = (y_2 + \frac{1}{2}y_1^2)e^{y_0},$$

$$A_3 = (y_3 + y_1y_2 + \frac{1}{3!}y_1^3)e^{y_0}$$

...

$$\tag{29}$$

from (28),(29) we get

$$y_0 = \frac{x^2}{25} + x^3 + \frac{x^5}{64} + \frac{x^8}{242} + \dots,$$

$$y_1 = \frac{-x^2}{25} - \frac{x^4}{1225} - \frac{x^5}{64} - \frac{x^6}{101250} - \frac{242}{89x^7} - \frac{23438x^8}{5671875} - \dots,$$

$$y_2 = \frac{-x^4}{1225} - \frac{74x^6}{2480625} - \frac{89x^7}{160000} - \frac{4622x^8}{7503890625} - \dots,$$

...

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 = x^3 - \frac{2x^4}{1225} - \frac{197x^6}{4961250} - \frac{89x^7}{80000} - \frac{10567x^8}{15007781250} - \dots$$

Table 1 and Figure 1 show comparison between the exact solution and MADM solution.

Table 1: Comparison of numerical errors between the exact solution $y = x^3$ and the MADM solution $y = \sum_{n=0}^2 y_n(x)$.

x	Exact	MADM	Absolut error
0.0	0.000	0.000000000	0.000000000
0.1	0.001	0.000999837	0.000000163
0.2	0.008	0.007997370	0.000002630
0.3	0.027	0.026986500	0.000013500
0.4	0.064	0.063956200	0.000043800
0.5	0.125	0.124889000	0.000111000
0.6	0.216	0.215755000	0.000245000
0.7	0.343	0.342512000	0.000488000
0.8	0.512	0.511087000	0.000913000
0.9	0.729	0.727375000	0.001625000
1.0	1.000	0.997214000	0.002786000

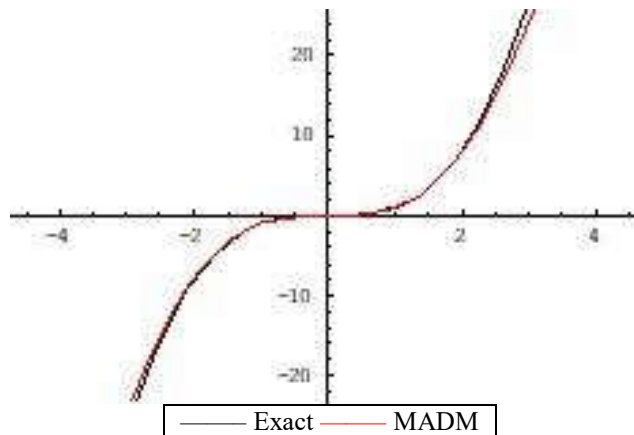


Figure 1: The exact solution $y = x^3$ and the MADM solution $y = \sum_{n=0}^2 y_n(x)$.

Case 2. when we put $n = m$, in eq.(1) we got multi-singular initial value problem, also from this equation we got different equations some of them have been studied while other equations are studied in this section.

$$y'' + \frac{2n}{x}y' + (k + \frac{n(n-1)}{x^2})y = g(x, y), \tag{30}$$

$$y(0) = a_0, y'(0) = a_1,$$

in an operator form eq.(30) become

$$Ly = g(x,y), \tag{31}$$

we proposed the differential operator

$$L(.) = \frac{1}{x^n \cos \sqrt{k}x} \frac{d}{dx} \cos^2 \sqrt{k}x \frac{d}{dx} \frac{x^n}{\cos \sqrt{k}x} (.), \tag{32}$$

and has an inverse L^{-1} which is a two-fold integral operator

$$L^{-1}(.) = \frac{\cos \sqrt{k}x}{x^n} \int_0^x \cos^{-2} \sqrt{k}x \int_0^x x^n \cos \sqrt{k}x (.) dx dx. \tag{33}$$

Applying the inverse operator L^{-1} to both side of eq.(31), we have

$$y(x) = \gamma(x) + L^{-1}(g(x,y)). \tag{34}$$

Now, we are going to give examples on eq.(30) and the equations which got them from eq.(30).

Problem 1. when $n = 2, k = 2$, in eq.(30) we obtain

$$y'' + \frac{4}{x}y' + (2 + \frac{2}{x^2})y = \frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x - y^2$$

$$y(0) = 0, y'(0) = 1, \tag{35}$$

noted that the exact solution is $y(x) = \sin x$.

Eq.(35) can be written as

$$Ly = \frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x - y^2, \tag{36}$$

where differential operator

$$L(.) = \frac{1}{x^2 \cos \sqrt{2}x} \frac{d}{dx} \cos^2 \sqrt{2}x \frac{d}{dx} \frac{x^2}{\cos \sqrt{2}x},$$

and inverse operator

$$L^{-1}(.) = \frac{\cos \sqrt{2}x}{x^2} \int_0^x \cos^{-2} \sqrt{2}x \int_0^x x^2 \cos \sqrt{2}x (.) dx dx.$$

On both sides of (36), and using the initial conditions at $x = 0$, yields

$$L^{-1}(Ly) = L^{-1}(\frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x) - L^{-1}y^2$$

$$y(x) = L^{-1}(\frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x) - L^{-1}y^2. \tag{37}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (37) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(\frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x) - L^{-1}y^2, \tag{38}$$

$$y_0 = L^{-1}(\frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x),$$

$$y_{n+1} = L^{-1}(\frac{4 \cos x}{x} + (1 + \frac{2}{x^2} + \sin x) \sin x) - L^{-1}A_n, n \geq 0, \tag{39}$$

where A_n are Adomian polynomials of nonlinear term y^2 , as below

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

$$A_2 = 2y_0y_2 + y_1^2, \quad (40)$$

Substituting (40) into (39) gives the components

$$y_0 = x - \frac{x^3}{6} + \frac{x^4}{30} + \frac{x^5}{120} - \frac{x^6}{140} - \frac{x^7}{5040} + \frac{37x^8}{56700} + \frac{x^9}{362880} - \frac{1961x^{10}}{113400} + \dots,$$

$$y_1 = \frac{-x^4}{30} + \frac{x^6}{140} - \frac{x^7}{1080} - \frac{x^8}{56700} + \frac{103x^9}{415800} + \frac{x^{10}}{113400} + \dots,$$

$$y_2 = \frac{-x^7}{1080} + \frac{103x^9}{415800} + \dots,$$

The solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{31x^7}{15120} + \frac{9943x^9}{19958400} + \dots \quad (41)$$

When we found y_0, y_1, y_2 , and collected them we noticed that, the solution by (MADM) approximated to exact solution. So, if we continue until y_n , we will get the exact solution.

Note that, the series of exact solution $y(x) = \sin x$ by Taylor series is as below

$$y(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \dots$$

Problem 2. In eq.(30) when $n = \frac{1}{2}$ and $k = 1$ we get the Bessel's equation of half order:

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = \frac{15}{4} + x^2 + e^{x^2} - e^y$$

$$y(0) = 0, y'(0) = 0. \quad (42)$$

Where $y(x) = x^2$ is exact solution.

Eq.(42) can be written as

$$Ly = \frac{15}{4} + x^2 + e^{x^2} - e^y, \quad (43)$$

where

$$L(.) = \frac{1}{\sqrt{x} \cos x} \frac{d}{dx} \cos^2 x \frac{d}{dx} \frac{\sqrt{x}}{\cos x} (.)$$

and operating

$$L^{-1}(.) = \frac{\cos x}{\sqrt{x}} \int_0^x \cos^{-2} x \int_0^x \sqrt{x} \cos x (.). dx dx.$$

On both sides of (43), and using the initial conditions at $x = 0$, yields

$$L^{-1}(Ly) = L^{-1}(\frac{15}{4} + x^2 + e^{x^2} - e^y)$$

$$y(x) = L^{-1}(\frac{15}{4} + x^2 + e^{x^2}) - L^{-1}e^y. \quad (44)$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (44) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(\frac{15}{4} + x^2 + e^{x^2}) - L^{-1}e^y. \quad (45)$$

$$y_0 = L^{-1}(\frac{15}{4} + x^2 + e^{x^2}),$$

$$y_{n+1} = -L^{-1}e^y, n \geq 0, \quad (46)$$

by using (46) we obtain

$$y_0 = \frac{19x^2}{15} + \frac{44x^4}{945} + \frac{1714x^6}{135135} + \frac{83234x^8}{34459425} + \frac{10820603x^{10}}{27498621150} + \dots,$$

$$y_1 = -L^{-1}(y_0) = \frac{-4x^2}{15} - \frac{4x^4}{63} - \frac{14842x^6}{675675} - \frac{5248786x^8}{861485625} - \frac{68638642333x^{10}}{43310328311250} - \dots,$$

$$y_2 = -L^{-1}(y_1) = \frac{-16x^4}{945} - \frac{7264x^6}{675675} - \frac{12891104x^8}{2584456875} - \frac{2782475392x^{10}}{1443677610375} - \dots,$$

The series solution is as below as

$$y(x) = y_0 + y_1 + y_2 = x^2 - \frac{32x^4}{945} - \frac{1504x^6}{75075} - \frac{22394912x^8}{2584456875} - \frac{67535227184x^{10}}{21655164155625} - \dots \tag{47}$$

Table 2 and Figure 2 show comparison between the exact solution and MADM solution.

Table 2: Comparison of numerical errors between the exact solution $y = x^2$ and the MADM solution $y = \sum_{n=0}^2 y_n(x)$.

x	Exact	MADM	Absolute error
0.00	0.0000	0.0000000000	0.0000000000
0.01	0.0001	0.0000999997	0.0000000003
0.02	0.0004	0.0003999950	0.0000000090
0.03	0.0009	0.0008999730	0.0000000270
0.04	0.0016	0.0015999100	0.0000000900
0.05	0.0025	0.0024997900	0.0000002100
0.06	0.0036	0.0035995600	0.0000004400
0.07	0.0049	0.0048991800	0.0000008200
0.08	0.0064	0.0063986100	0.0000013900
0.09	0.0081	0.0080977700	0.0000022300
0.1	0.0100	0.0099965900	0.0000034100

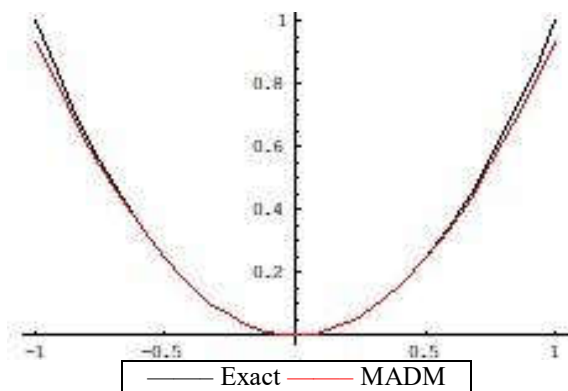


Figure 2: The exact solution $y = x^2$ and the MADM solution $y = \sum_{n=0}^2 y_n(x)$.

IV. CONCLUSION

A modified Adomian decomposition method (MADM) was prepared to solve multi singular initial value problem in the second-order differential equation like Bessel’s equation of half order by using a new differential operator. We applied this method on these equations and we compared the solutions with the exact solution by using the tables and diagrams, we found that the solution by (MADM) are approaching the exact solution of small duplicates. So, if we continue until y_n , we will get the exact solution. The study also shows that the (MADM) is an applicable and powerful method to solve problems considered.

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