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NEW DEVELOPMENT OF ADOMIAN DECOMPOSITION METHOD FOR SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract:-
This study attempts at Adomian Decomposition Method (ADM) for solving second order ordinary differential equations such as Multi singular equation, Bessel's equation, and Oscillatory systems. The (ADM) results in many known equations, some of them are studied and investigated by mathematicians and researchers, while others are still not well researched yet. We give a new differential operator and invers differential operator to solve different types for initial value problem in the second order ordinary differential equation. There are some different situations that we will study and give the non-linear examples that lead us to the approximate solutions of exact solution.

Keywords:-*Adomian decomposition method, Second order ordinary differential equation, Singular initial value problems, Multi singular equation.*

I. INTRODUCTION

We suppose the second order ordinary differential equation as the form:

$$y'' + ((m+n) + \frac{a+b}{x})y' + (nm + \frac{nb+am}{x} + \frac{a(b-1)}{x^2})y = f(x,y), \quad (1)$$

With initial conditions

$$y\ (0) = a, y^0(0) = b,$$

Where $f(x,y)$ is given function, a, b, m, and n are constants.

This type of equation has great importance and necessity, because it produces some famous equations and it is studyied by some scientists and researchers, like Multi singular equation [10], Oscillatory systems [11], and Emden-Fowler and Lane-Emden type equation [8]. In addition, it has wide applications in math, enginering, mechanics and other science. Many researchers studied singular initial value problems such as Cui M. and Geng F. Solving singular two-point boundary value problem in reproducing kernel space [7], and Tawfiq L.N.M and Hussein R.W. On solution of regular singular initial value problems [6].

The Adomian decomposition manner was introduced by Grorge Adomian in (1970) [3]. This method is used widely for solving non-linear ordinary differential equation [5]. The solution get by this manner has a series shape which convergent and easy accounting process. When we write the non-linear term like a series of polynomials, we can get a series of solutions which are called Adomian polynomials. The (ADM) give importance to the study of singular value problem by many researchers for example Wazwaz A.M. A new method for solving initial value problems in the second order ordinary differential equations [1], and Hasan Y.Q. and Zhu L.M. singular boundary value problems of higher ordinary differential equation by modified Adomian decomposition method [9]. In this research we introduce solutions for many singular initial value problems of second order ordinary differential equation by using (ADM) to give a new differential operator which works to solve these equations in a smooth and good way which leads us to the approximate solutions from the exact solution, or may be exactly the exact solution.

II. Adomian decomposition method

We suggest a new differential operator as below:

$$L(.) = x^{-b}e^{-mx}\frac{d}{dx}x^{b-a}e^{(m-n)x}\frac{d}{dx}x^{a}e^{nx}(.), \quad (2)$$

We can write the equation (1), as follows

$$Ly = f(x,y). \quad (3)$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator as follows

$$L^{-1}(.) = x^{-a}e^{-nx}\int_0^x x^{a-b}e^{(n-m)x}\int_0^x x^{b}e^{mx}(.)dxdx, \quad (4)$$

By taking L^{-1} in both parts equation (3), we have

$$Y(x) = \varphi(x) + L^{-1}f(x,y). \quad (5)$$

The Adomian Decomposition Manner introduce the solution $y(x)$ and the non-linear function $f(x,y)$ by infinite series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (6)$$

And

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \quad (7)$$

Where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [2,4] to formulate Adomian polynomials. The following algorithm:

$$A_0 = f(y_0),$$

$$A_1 = y_1 f'(y_0),$$

$$A_2 = y_2 f'(y_0) + \frac{1}{2}y_1^2 f''(y_0),$$

$$A_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!}y_1^3 f'''(y_0)$$

$$\vdots \quad (8)$$

Can be used to construct Adomian polynomilas, when $f(y)$ is a non-linear function. By substituting (6) and (7) into (5), we get

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + L^{-1}\sum_{n=0}^{\infty} A_n. \qquad (9)$$

Through using Adomian Decomposition Manner, the components $y(x)$ can be determined as

$$y_0 = \varphi(x),$$

$$y_{n+1} = L^{-1}A_n, n \geq 0, \qquad (10)$$

Which gives

$$y_1 = L^{-1}A_0,$$
$$y_2 = L^{-1}A_1,$$
$$y_3 = L^{-1}A_2, \qquad (11)$$
$$\vdots$$

From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (6) can be immediately obtained.

$$\phi_n(x) = \sum_{i=0}^{n-1} yi, \qquad (12)$$

it can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial valve problems.

III. NUMERICAL APPLICATIONS AND THE DISCUSSIONS OF DFFERENT CASES.

We will give illustrative examples for non-linear equations by (ADM) and explain these cases as follows

Case 1. We will give one example to explain Eq.(1).

Example 1.1.

We will explain this case by example which shows the convergence of the solution, by put $m = 2, n = 3, a = 1, b = 2,$ in equation(1), as follows:

$$y'' + ((5 + \frac{3}{x})y' + (6 + \frac{8}{x} + \frac{1}{x^2}) = (12 + \frac{11}{x} + \frac{1}{x^2})e^x + x - \ln y, \qquad (13)$$

$$y(0) = 1, y^0(0) = 1,$$

with exact solution $y(x) = e^x$.

The differential operator L for this equation as below

$$L(.) = x^{-2}e^{-2x}\frac{d}{dx}xe^{-x}\frac{d}{dx}xe^{3x}(.), \qquad (14)$$

and the invers differential operator L^{-1} for this equation as below

$$L^{-1}(.) = x^{-1}e^{-3x}\int_0^x x^{-1}e^x\int_0^x x^2e^{2x}(.)dxdx, \qquad (15)$$

in an operator form, Eq.(13), yield to

$$Ly = (12 + \frac{11}{x} + \frac{1}{x^2})e^x + x - \ln y, \qquad (16)$$

by taken L^{-1} in both parts of (16), we have

$$y = L^{-1}((12 + \frac{11}{x} + \frac{1}{x^2})e^x + x) - L^{-1}\ln y. \qquad (17)$$

goning on as before we get, the recursive relationship:

$$y_0 = L^{-1}((12 + \frac{11}{x} + \frac{1}{x^2})e^x + x),$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0. \qquad (18)$$

The Adomian polynomials for the non-linear part $f(y) = \ln y,$ as follows

$$A_0 = \ln y_0,$$

$$A_1 = \frac{y_1}{y_0},$$

$$A_2 = \frac{y_2}{y_0} - \frac{y_1^2}{y_0^2},$$

Thus, the first few components are as below:

$$y_0 = 1+x+\frac{x^2}{2}+\frac{11\,x^3}{48}-\frac{19\,x^4}{1200}+\frac{307\,x^5}{7200}-\frac{5177\,x^6}{352800}+\frac{18401\,x^7}{2822400}-\frac{163159\,x^8}{76204800}+\frac{168437\,x^9}{254016000}+...,$$

$$y_1 = \frac{-x^3}{16}+\frac{23\,x^4}{400}-\frac{173\,x^5}{4800}+\frac{1543\,x^6}{78400}-\frac{115487\,x^7}{11289600}+\frac{933131\,x^8}{182891520}-\frac{5285309\,x^9}{2177280000}+...,$$

$$y_2 = \frac{x^5}{576}-\frac{851\,x^6}{235200}+\frac{59239\,x^7}{15052800}-\frac{741529\,x^8}{243855360}+\frac{8025587\,x^9}{4064256000}-...,$$

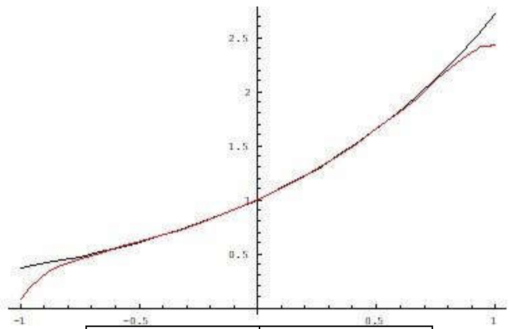
Therefore

$$y(x) = y_0 + y_1 + y_2 =$$

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{97\,x^7}{430080}-\frac{291947\,x^8}{3657830400}+\frac{12820033\,x^9}{60963840000}-...$$

Table 1. Difference between ADM and exact solution

x	Exact	ADM	Error
0	1.0000	1.0000	0000
0.1	1.10517	1.10517	00000
0.2	1.2214	1.2214	00000
0.3	1.34986	1.34986	00000
0.4	1.49182	1.49179	0.00003
0.5	1.64872	1.64844	0.00028
0.6	1.82212	1.82037	0.00175
0.7	2.01375	2.00559	0.00816
0.8	2.22554	2.1945	0.03104
0.9	2.4596	2.3588	0.1008



—— Exact ——	ADM

Figure 1. The Approximation for ADM and Exact solution

Tablel 1. Displays the comparison betweeen results of the Adomian decomposition method and the exact solutions calculate for some chosen values. We can get from figure 1. That the (ADM) is correct, more active and converges to the exact solution.

Case 2. When, n=m in Eq.(1).

The resulting ordinary differential equation gives the following form:

$$y'' + (2n + \frac{a+b}{x})y' + (n^2 + \frac{n(a+b)}{x} + \frac{a(b-1)}{x^2})y = f(x,y)$$

$$y(0) = a, y'(0) = b. \quad , \qquad (19)$$

Where $f(x,y)$ is given function, n, a and b are constants.

The differential operator L for this equation as below:

$$L(.) = x^{-b}e^{-nx}\frac{d}{dx}x^{b-a}\frac{d}{dx}x^{a}e^{nx}(.) \qquad (20)$$

so, the invers differential operator L^{-1} for this equation as form:

$$L^{-1}(.) = x^{-a}e^{-nx}\int_0^x x^{a-b}\int_0^x x^{b}e^{nx}(.)dxdx. \qquad (21)$$

In general, the general solution for this equation gives the following:

$$y(x) = \varphi(x) + L^{-1}f(x,y). \qquad (22)$$

To explain this case, we will give example as below:

Example 2.1.
We will put $n = 2, a = 1, b = 2,$ in Eq.(19), we have

$$y'' + (4 + \frac{3}{x})y' + (4 + \frac{6}{x} + \frac{1}{x^2}) = x^8 + 4x^2 + 14x + 9 - y^4, \qquad (23)$$

with initial conditions $y(0) = 0, y^0(0) = 0$.
The differential operator L for this equation as below:

$$L(.) = x^{-2}e^{-2x}\frac{d}{dx}x\frac{d}{dx}xe^{2x}(.), \qquad (24)$$

so, the invers differential operator L^{-1} as form:

$$L^{-1}(.) = x^{-1}e^{-2x}\int_0^x x^{-1}\int_0^x x^{2}e^{2x}(.)dxdx. \qquad (25)$$

In general, the general solution for this equation gives the following form:

$$y(x) = L^{-1}(x^8 + 4x^2 + 14x + 9) - L^{-1}(y^4), \qquad (26)$$

where that $y_0 = L^{-1}(x^8 + 4x^2 + 14x + 9)$, $yn+1 = -L-1An, n \geq 0$.
The Adomian polynomials for the non-linear part $f(y) = y^4$ as follows

$$A_0 = y_0^4,$$

$$A_1 = 4y_0^3y_1,$$

$$A_2 = 4y_0^3y_2 + 6y_0^2y_1^2,$$

so

$$y_0 = x^2 + \frac{x^{10}}{121} - \frac{23\,x^{11}}{8712} + \frac{431\,x^{12}}{736164} - \frac{3875\,x^{13}}{36072036} + \frac{7793\,x^{14}}{450900450} - \frac{12059\,x^{15}}{4809604800} + \ldots,$$

$$y_1 = \frac{-x^{10}}{121} + \frac{23\,x^{11}}{8712} - \frac{431\,x^{12}}{736164} + \frac{3875\,x^{13}}{36072036} - \frac{7793\,x^{14}}{450900450} + \frac{12059\,x^{15}}{4809604800} + \ldots$$

Then, from y_0 and y_1, we can get the exact solution $y(x) = x^2$, by this method.

Case 3. When n=m, a=0, in Eq.(1).
The resulting ordinary differential equation give the following form:

$$y'' + (2n + \frac{b}{x})y' + (n^2 + \frac{nb}{x})y = f(x,y)$$

$$y(0) = a, y'(0) = b. \qquad (27)$$

Where $f(x,y)$ is given function, n and b are constants. The differential operator L for this equation is:

$$L(.) = x^{-b}e^{-nx}\frac{d}{dx}x^{b}\frac{d}{dx}e^{nx}(.), \qquad (28)$$

and, the invers differential operator L^{-1} for this equation is:

$$L^{-1}(.) = e^{-nx}\int_0^x x^{-b}\int_0^x x^{b}e^{nx}(.)dxdx. \qquad (29)$$

The general solution for this equation given by:

$$y(x) = y(0)e^{-nx} + L^{-1}f(x,y). \qquad (30)$$

We will give example which will illustrate this case, as following:

Example 3.1.
put $n = 3, b = 2$. Eq.(27), we get

$$y'' + (6 + \frac{2}{x})y' + (9 + \frac{6}{x})y = (25 + \frac{10}{x})e^{2x} + 4x - \ln y^2, \qquad (31)$$

$$y(0) = 1, y'(0) = 2,$$

with exact solution $y(x) = e^{2x}$, the differential operator L yield to

$$L(.) = x^{-2}e^{-3x}\frac{d}{dx}x^{2}\frac{d}{dx}e^{3x}(.), \qquad (32)$$

so, the invers differential operator L^{-1} becomes

$$L^{-1}(.) = e^{-3x}\int_0^x x^{-2}\int_0^x x^2 e^{3x}(.)dxdx, \quad (33)$$

we can rewrite the equation (31) as below

$$Ly = (25+\frac{10}{x})e^{2x} + 4x - L^{-1}(\ln y^2), \quad (34)$$

by taking invers differential operator L^{-1} of both parts (34), we have

$$y(x) = e^{-3x} + L^{-1}(25+\frac{10}{x})e^{2x} + 4x - L^{-1}(\ln y^2),$$

using the decomposition series for the linear function and the polynomial series for the non-linear part, we get the recursive relationship

$$y_0 = e^{-3x} + L^{-1}(25+\frac{10}{x})e^{2x} + 4x,$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0. \quad (35)$$

The Adomian polynomials for the non-linear part $f(y) = \ln y^2$ are computed as below:

$$A_0 = \ln y_0^2,$$

$$A_1 = \frac{2y_1}{y_0},$$

$$A_3 = \frac{2y_2}{y_0} - \frac{y_1^2}{y_0^2}, \quad (36)$$

we will put (36) in (35), we obtain the components

$$y_0 = 2x+2x^2+\frac{5x^3}{3}+\frac{4x^4}{15}+\frac{17x^5}{30}-\frac{26x^6}{315}+\frac{533x^7}{5040}-\frac{13x^8}{504}+\frac{5743x^9}{453600}-\frac{2011x^{10}}{623700}+\ldots,$$

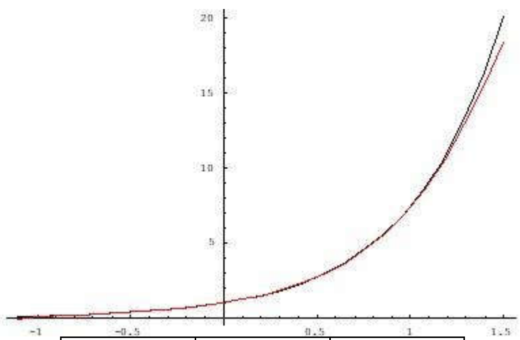
$$y_1 = \frac{-x^3}{3}+\frac{2x^4}{5}-\frac{29x^5}{90}+\frac{76x^6}{315}-\frac{323x^7}{1680}+\frac{7x^8}{45}-\frac{55129x^9}{453600}+\frac{407873x^{10}}{4158000}+\ldots,$$

$$y_2 = \frac{x^5}{45}-\frac{22x^6}{315}+\frac{71x^7}{630}-\frac{1459x^8}{11340}+\frac{28993x^9}{226800}-\frac{39889x^{10}}{297000}+\ldots,$$

then, the general solution given by $y(x) = y_0+y_1+y_2 = 1.+2.x+2.x^2+1.33333x^3+0.666667x^4+0.266667x^5 +0.0888889x^6+0.0261905x^7+0.00110229x^8+0.0189594x^9-0.0394371x^{10}+\ldots$

Table 2. Difference between ADM and exact solution

x	Exact	ADM	Error
0	1.0000	1.0000	0000
0.1	1.2214	1.2214	00000
0.2	1.49182	1.49182	00000
0.3	1.82212	1.82212	00000
0.4	2.22554	2.22554	0.00000
0.5	2.71828	2.71826	0.00002
0.6	3.32012	3.31999	0.00013
0.7	4.0552	4.05455	0.00065
0.8	4.95303	4.9504	0.00263
0.9	6.04965	6.0407	0.00895



—	Exact —	ADM

Figure 2. The Approximation for ADM and Exact solution

Table 2. Displays the comparison betweeen results of the Adomian decomposition method and the exact solutions calculate for some chosen values. We can get from figure 2. that the (ADM) is correct, more active and converges to the exact solution.

case 4. when n=m, a=-b, in Eq.(1).
The resulting ordinary differential equation gives the following form:

$$y'' + 2ny' + (n^2 - \frac{b(b-1)}{x^2})y = f(x,y)$$

$$y(0) = a, y'(0) = b. \quad , \qquad (37)$$

The differential operator L for this equation as below:

$$L(.) = x^{-b}e^{-nx}\frac{d}{dx}x^{2b}\frac{d}{dx}x^{-b}e^{nx}(.) \quad , \qquad (38)$$

and, the invers differential operator L^{-1} for this equation as below:

$$L^{-1}(.) = x^{b}e^{-nx}\int_0^x x^{-2b}\int_0^x x^{b}e^{nx}(.)dxdx. \qquad (39)$$

The general solution given by:

$$y(x) = \varphi(x) + L^{-1}f(x,y). \qquad (40)$$

For this case, we will give one example as the following.

Example 4.1.
Suppose $b = 2, n = 0$, in Eq.(37), the equation yiled to

$$y'' - \frac{2}{x^2}y = 4x + x^6 - y^2$$

$$y(0) = 0, y'(0) = 0, \quad , \qquad (41)$$

with exact soulion $y(x) = x^3$. The differential operator L becomes

$$L(y) = x^{-2}\frac{d}{dx}x^4\frac{d}{dx}x^{-2}(y),$$

so

$$L^{-1}(Ly) = x^2\int_0^x x^{-4}\int_0^x x^2(y'' - \frac{2}{x^2}y)dxdx.$$

In a differential operator form, Eq.(41), becomes

$$Ly = 4x + x^6 - y^2. \qquad (42)$$

Applying L^{-1} on both parts of(42), we get

$$y(x) = L^{-1}(4x + x^6) - L^{-1}y^2,$$

so

$$y_0 = L^{-1}(4x + x^6),\ yn+1 = yn2,$$

where the Adomian polynomials for $f(y) = y^2$, are

$$A_0 = f(y_0) = y_0^2,$$

$$A_1 = y_1 f'(y_0) = 2y_0y_1,$$

$$A_2 = y_2 f'(y_0) + \frac{1}{2}f''(y_0)y_1^2 = 2y_0y_2 + y_1^2,$$

so

$$y_0 = x^3 + \frac{x^8}{54},$$

$$y_1 = \frac{-x^8}{54} - \frac{x^{13}}{4158} - \frac{x^{18}}{886464},$$

$$y_2 = \frac{4678756992\,x^{13} + 74673144\,x^{18} + 430911\,x^{23} + 1078\,x^{28}}{19454271572736}$$

$$y(x) = y_0 + y_1 + y_2 =$$

$$x^3 + \frac{185\,x^{18}}{68257728} + \frac{127\,x^{23}}{5733649152} + \frac{x^{28}}{18046634112}. \quad ,$$

Table 3. Difference between ADM solution and exact solution

x	Exact	ADM	Error
0.0	0.000	0.000	0000
0.1	0.001	0.001	0000
0.2	0.008	0.008	0000
0.3	0.027	0.027	0000
0.4	0.064	0.064	0000
0.5	0.125	0.125	0000
0.6	0.216	0.216	0000
0.7	0.343	0.343	0000
0.8	0.512	0.512	0000
0.9	0.729	0.729	0000
1.0	1.000	1.000	0000

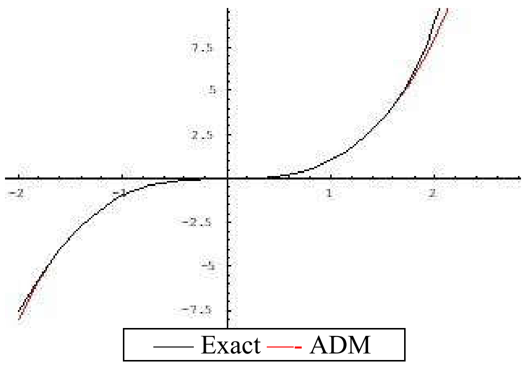


Figure 3. The Approximation for ADM and Exact solution

Taple 3. Displays the comparison betweeen results of the Adomian decomposition method and the exact solutions calculate for some chosen values. We can get from figure 3. that the (ADM) is correct, more active and converges to the exact solution.

Case 5. when a=b, in Eq.(1).

The resulting differential equation is:

$$y'' + ((m+n) + \frac{2b}{x})y' + (mn + \frac{b(m+n)}{x} + \frac{b(b-1)}{x^2})y = f(x,y), \qquad (43)$$

with initial conditions $y(0) = a, y^0(0) = b$.

The differential operator L for this equation is taking a shape:

$$L(.) = x^{-b}e^{-mx}\frac{d}{dx}e^{(m-n)x}\frac{d}{dx}x^b e^{nx}(.), \qquad (44)$$

so, the invers differential operator L^{-1} is taking a shape:

$$L^{-1}(.) = x^{-b}e^{-nx}\int_0^x e^{(n-m)x}\int_0^x x^b e^{mx}(.)dxdx. \qquad (45)$$

In general, the general solution for this equation as below:

$$y(x) = \varphi(x) + L^{-1}f(x,y).$$

To illustrate this situation, we will give the following example

Example 5.1.

We will put $m = 1, n = -3, b = 0,$ in Eq.(43), we get

$$y'' - 2y' - 3y = -4e^x + 2e^{2x} - 2yy', \qquad (46)$$
$$y(0) = 1, y'(0) = 1,$$

the exact soluion $y(x) = e^x$.

$$L(.) = e^{-x}\frac{d}{dx}e^{4x}\frac{d}{dx}e^{-3x}(.), \qquad (47)$$

and, the invers differential operator L^{-1} defined by:

$$L^{-1}(.) = e^{3x}\int_0^x e^{-4x}\int_0^x e^x(.)dxdx, \qquad (48)$$

We can rewrite (46) as follows

$$Ly = -4e^x + 2e^{2x} - 2yy', \tag{49}$$

By take L^{-1} on both parts of (49), we get

$$y(x) = L^{-1}(-4e^x + 2e^{2x}) - 2L^{-1}(yy'), \tag{50}$$

pursuit as before we have the recursive relationship

$$y_0(x) = \frac{1}{2}e^{3x} + \frac{1}{2}e^{-x} + L^{-1}(-4e^x + 2e^{2x})$$

$$y_{n+1} = -2L^{-1}(y_n y_n'), n \geq 0, \tag{51}$$

then, the first few components are as below:

$$y_0 = 1+x+\frac{3x^2}{2}+\frac{3x^3}{2}+\frac{31x^4}{24}+\frac{101x^5}{120}+\frac{323x^6}{720}+\frac{1009x^7}{5040}+\frac{1037x^8}{13440}+\frac{3167x^9}{120960} + \frac{28843x^{10}}{3628800}+\frac{87209x^{11}}{39916800}+\frac{262991x^{12}}{479001600}+\frac{791701x^{13}}{6227020800}+\ldots,$$

$$y_1 = -x^2-2x^3-\frac{11x^4}{4}-\frac{89x^5}{30}-\frac{109x^6}{40}-\frac{563x^7}{252}-\frac{33601x^8}{20160}-\frac{11497x^9}{10080}-\frac{48143x^{10}}{67200} - \frac{4121087x^{11}}{9979200}-\frac{17456603x^{12}}{79833600}-\frac{165974377x^{13}}{1556755200}-\ldots,$$

$$y_2 = \frac{-x^5}{5}-\frac{11x^6}{15}-\frac{166x^7}{105}-\frac{2143x^8}{840}-\frac{10273x^9}{3024}-\frac{10619x^{10}}{2700}-\frac{1698869x^{11}}{415800} - \frac{19396339x^{12}}{4989600}-\frac{80994211x^{13}}{23587200}-\ldots,$$

therefore

$$y(x) = y_0 + y_1 + y_2 =$$

$$1+x+\frac{x^2}{2}-\frac{x^3}{2}-\frac{35x^4}{24}-\frac{93x^5}{40}-\frac{2167x^6}{720}-\frac{6073x^7}{1680}-\frac{33391x^8}{8064}-\frac{545717x^9}{120960} - \frac{3368563x^{10}}{725760}-\frac{59829521x^{11}}{13305600}-\frac{1966525171x^{12}}{479001600}-\frac{7348525837x^{13}}{2075673600}-\ldots$$

Table 4. Difference between ADM solution and exact solution

x	Exact	ADM	Error
0.0	1.0000	1.0000	0000
0.1	1.10517	1.10433	0.00084
0.2	1.2214	1.21267	0.00873
0.3	1.34986	1.31065	0.03921
0.4	1.49182	1.36393	0.12789
0.5	1.64872	1.49004	0.15868

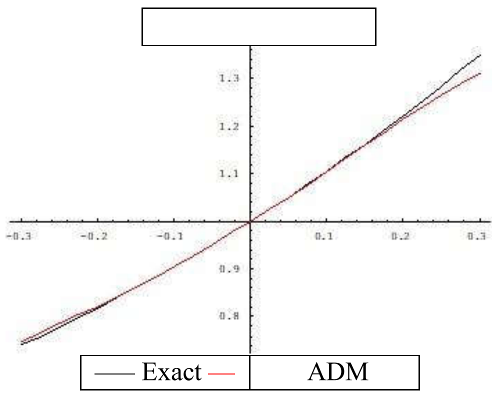


Figure 4. The Approximation for ADM and Exact solution

Table 4. Displays the comparison betweeen results of the Adomian decomposition method and the exact solutions calculate for some chosen values. We can get from figure 4. that the (ADM) is correct, more active and converges to the exact solution.

Case 6.when n=0, in Eq(1). Then, the resulting differential equation is:

$$y'' + (m + \frac{a+b}{x})y' + (\frac{ma}{x} + \frac{a(b-1)}{x^2})y = f(x,y), \quad (52)$$

The differential operator L of this equation is taking a shape:

$$L(.) = x^{-b}e^{-mx}\frac{d}{dx}x^{b-a}e^{mx}\frac{d}{dx}x^{a}(.), \quad (53)$$

so, the invers differential operator L^{-1} is taking a shape:

$$L^{-1}(.) = x^{-a}\int_0^x x^{a-b}e^{-mx}\int_0^x x^b e^{mx}(.)dxdx. \quad (54)$$

In general, the general solution for this equation as below:

$$y(x) = \varphi(x) + L^{-1}f(x,y).$$

For this case, we will introduce as below

Example 6.1.

When we put $m = 0, b = 1, a = 3$, in Eq(52). we get

$$y'' + \frac{4}{x}y' = e^x(1 + \frac{4}{x}) + e^{2x} - y^2, \quad (55)$$

with initial condition $y(0) = 1, y^0(0) = 1$.

The diffrerntial operator L for this equation:

$$L(.) = x^{-1}\frac{d}{dx}x^{-2}\frac{d}{dx}x^3(.), \quad (56)$$

and, the invers diffrerntial operator L^{-1} as below:

$$L^{-1}(.) = x^{-3}\int_0^x x^2\int_0^x x(.)dxdx, \quad (57)$$

goning on as before we get the recursive relationship

$$y_0 = 1 + L^{-1}e^x(1 + \frac{4}{x}) + e^{2x},$$

$$yn+1 = -L-1An, n \geq 0. \quad (58)$$

The Adomian polynomials for the non-linear part $f(y) = y^2$, as follows

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

$$A_2 = y_1^2 + 2y_0y_2,$$

Thus, the first little components are as below:

$$y_0 = 1+x+\frac{3x^2}{5}+\frac{5x^3}{18}+\frac{19x^4}{168}+\frac{x^5}{24}+\frac{89x^6}{6480}+\frac{101x^7}{25200}+\frac{51x^8}{49280}+\frac{37x^9}{155520}+\frac{331x^{10}}{6739200}+\ldots,$$

$$y_1 = \frac{-x^2}{10}-\frac{x^3}{9}-\frac{11x^4}{140}-\frac{79x^5}{1800}-\frac{7193x^6}{340200}-\frac{9x^7}{980}-\frac{2447x^8}{665280}-\frac{16819x^9}{12247200}-\frac{363859x^{10}}{756756000}+\ldots,$$

$$y_2 = \frac{x^4}{140}+\frac{19x^5}{1800}+\frac{1573x^6}{170100}+\frac{911x^7}{147000}+\frac{105019x^8}{29937600}+\frac{2267537x^9}{1285956000}+\frac{1413971x^{10}}{1746360000}+\ldots,$$

the general solution

$$y(x) = y_0+y_1+y_2 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{139x^6}{75600}+\frac{901x^7}{882000}+\frac{51773x^8}{59875200}+\frac{3229943x^9}{5143824000}+\frac{34323637x^{10}}{90810720000}+\ldots$$

Table 5. Difference between ADM solution and exact solution

x	Exact	ADM	Error
0.0	1.0000	1.0000	0000
0.1	1.10517	1.10517	00000
0.2	1.2214	1.2214	00000
0.3	1.34986	1.34986	00000
0.4	1.49182	1.49183	0.00001
0.5	1.64872	1.64874	0.00002
0.6	1.82212	1.82219	0.00007
0.7	2.01375	2.01396	0.00021
0.8	2.22554	2.2261	0.00056
0.9	2.4596	2.46097	0.0013

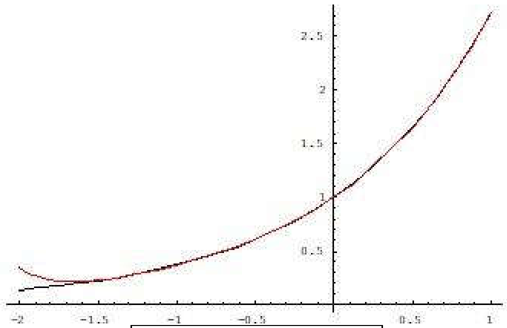


Figure 5. The Approximation for ADM and Exact solution

Table 5. Displays the comparison betweeen results of the Adomian decomposition method and the exact solutions calculate for some chosen values. We can get from figure 5. that the (ADM) is correct, more active and converges to the exact solution.

Case 7. When a=b, n=-m, in Eq(1).

The resulting differential equation is:

$$y'' + \frac{2b}{x}y' + (-m^2 + \frac{b(b-1)}{x^2})y = f(x,y), \tag{59}$$

with the initial conditions

$$y(0) = a, y'(0) = b.$$

The differential operator L for this equation is:

$$L(.) = x^{-b}e^{-mx}\frac{d}{dx}e^{2mx}\frac{d}{dx}x^b e^{-mx}(.), \tag{60}$$

we can rewrite equation (59) as below

$$Ly = f(x,y), \tag{61}$$

and, the invers differential operator L^{-1} for this equation is:

$$L^{-1}(.) = x^{-b}e^{mx}\int_0^x e^{-2mx}\int_0^x x^b e^{mx}(.)dxdx, \tag{62}$$

by taken the invers differential operator L^{-1} in both parts(61), we have

$$y(x) = \varphi(x) + L^{-1}f(x,y).$$

For this case, we will give illustrate example as follows.

Example 7.1

When we put $m = \sqrt{-1} = i$ and $b = \frac{1}{2}$ in Eq.(59), we get:

$$y'' + \frac{1}{x}y' + (1 - \frac{\frac{1}{4}}{x^2})y = \frac{15}{4} + x^2 + x^4 - y^2, \tag{63}$$

This equation, is Bessel's equation of order $\frac{1}{2}$.

The differential operator L for this equation as below:

$$L(.) = x^{-1}e^{ix}\frac{d}{dx}e^{-2ix}\frac{d}{dx}xe^{ix}(.), \tag{64}$$

so, the invers differential operator L^{-1} for this equation is:

$$L^{-1}(.) = x^{-1}e^{-ix}\int_0^x e^{2ix}\int_0^x xe^{-ix}(.)dxdx, \quad (65)$$

we can rewrite the Eq.(63), we have

$$Ly = \frac{15}{4} + x^2 + x^4 - y^2, \quad (66)$$

by taking invers differential operator L^{-1} in both parts(66), we have

$$y(x) = L^{-1}(\frac{15}{4} + x^2 + x^4 - y^2) - L^{-1}y^2, \quad (67)$$

so, the recursive relation we get

$$y_0(x) = \frac{15}{4} + x^2 + x^4 - y^2,, \quad (68)$$

And

$$y_{n+1}(x) = -L^{-1}y^2. \quad (69)$$

The Adomian polynomials for the non-linear part $f(y) = y^2$, as follows

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

$$A_2 = y_1^2 + 2y_0y_2,$$

consequently, the first components are as below:

$$y_0 = x^2 + \frac{4\,x^4}{63} + \frac{4\,x^6}{143,}$$

$$y_1 = \frac{4\,x^6}{143} + \frac{32\,x^8}{16065} + \frac{136160\,x^{10}}{226459233} + \frac{128\,x^{12}}{5180175} + \frac{64\,x^{14}}{16011567,}$$

$$y_2 = \frac{32\,x^{10}}{57057} + \frac{69248\,x^{12}}{1320944625} + \frac{1231670144\,x^{14}}{79825747336335} + \frac{132321491968\,x^{16}}{142666820076956175} + \frac{96699877888\,x^{18}}{699297024827275875} + \frac{4384768\,x^{20}}{927450404305425} + \frac{2048\,x^{22}}{4430480646735}.$$

The approximant is this as below:

$$y(x) = y_0 + y_1 + y_2 =$$

$$x^2 + \frac{4\,x^4}{63} + \frac{8\,x^6}{143} + \frac{32\,x^8}{16065} + \frac{263168\,x^{10}}{226459233} + \frac{101888\,x^{12}}{1320944625} + \frac{1550742464\,x^{14}}{79825747336335} + \frac{132321491968\,x^{16}}{142666820076956175} + \frac{96699877888\,x^{18}}{699297024827275875} + \frac{4384768\,x^{20}}{927450404305425} + \frac{2048\,x^{22}}{4430480646735}$$

.

Table 6. Difference between ADM solution and exact solution

x	Exact	ADM	Error
0	00001	00001	0000
0.1	0.01	0.0100064	0.0000064
0.2	0.04	0.0401052	0.000105
0.3	0.09	0.090552	0.000552
0.4	0.16	0.161856	0.00001
0.5	0.25	0.254851	0.004851

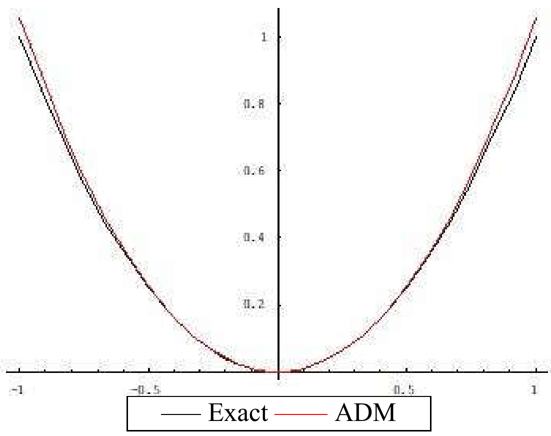


Figure 6. The Approximation for ADM and Exact solution

Table 6. Displays the comparison between results of the Adomian decompo sition method and the exact solutions calculate for some chosen values. We can look from figure 6. that the (ADM) is correct, more active and converges to the exact solution.

IV. CONCLUSION

Adomian decomposition method (ADM) is an effective organizational methed for solving singular and nonsingular initial value problem of second order ordinary differential equations. We have explained a new differential operator for solving these equations such as Multi singular equation, Bessel's equation and Oscillatory systems. We will give different cases and illustrative examples (1-7), to explain this method. (ADM) is powerfule, active, more accurate, so it leads us to approximate solution from exact solution. Also it has high efficiency of initial value problem.

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