EPH - International Journal of Mathematics and Statistics
ISSN (Online): 2208-2212
Volume 2 Issue 2 July 2016

DOI:https://doi.org/10.53555/eijms.v6i1.46

APPROXIMATE SOLUTIONS TO BOUNDARY VALUE PROBLEMS OF HIGHER-ORDER BY THE MODIFIED ADOMAIN DECOMPOSITION METHOD

Zainab Ali Abdu AL-Rabahi[2], Yahya Qaid Hasan[1*]
[2]*Department of Mathematics Sheba Region University, Yemen*
[*1]*Department of Mathematics Sheba Region University, Yemen*

Corresponding Author:-

Abstract*:-*

This paper propose a new modified differential operator for solving third-order boundary value problems into higherorder ordinary differential equation. We found the differential operator for new two inverse operators which can be applied for solving equations at more than one type in different conditions. This paper is given five non-linear examples from a high-order, we got effective and easily approximate solutions to the exact solution.

Keywords*:-*"*Boundary Value Problem, Adomain Decomposition Method, Boundary Conditions, Higher-Order non-linear ODE.*"

I. INTRODUCTION

In this research, we will introduce Boundary Value Problems of the form

$$y^{(n+2)} = f(x,y,y',...,y^{(n+1)}), n \geq 1, \tag{1}$$

With one of the following conditions

$$y(0) = c_0, y'(0) = c_1,...,y^{(n)}(0) = c_n, y^{(n-m)}(b) = d, \tag{2}$$

$$y(a_0) = b_0, y'(a_1) = b_1,...,y^{(n-m)}(a_n) = b_n, y^{(n-m)}(0) = h, y^{(n)}(0) = d_1. \tag{3}$$

Where f is a differential operator of linear or non-linear order less than $(n+2)$ and $m = 0$ or $m = 1$, and $a_0,a_1,...,a_n,c_0,c_1,...,c_n,h,d,d_1,b,b_0,b_1,...,b_n$ are constants.

Numerical solutions of higher order boundary value problems has not re ceived a full concern of research as in literature [1, 8, 9, 14, and 15]. There are many evidences that indicate to the presence and distinctiveness of these problems which deserve to be subjected to a further studies as it is shown clearly by [13]. According to Dirichlet, Neumann or Robin states or circumstances of the modified decomposition method, numerous kinds of boundary value problems were solved through many works by Adomain [11, 12] as well as Adomain and Rach [10]. Numerical and analytical solutions of Breatu equa tion were actually attained when Deeba et al [8] had used Adomain method. The solution of boundary value problems with Dirichlet and Neumann states or circumstances were obtained through making use of Adomain method by Wazwaz [3,6]. In addition, certain solutions for non-linear boundary value problems were come into existence by a trustworthy algorithm offered by Wazwaz [4]. The achievement of the flow up solutions by mixed boundary circumstances has been Wazwaz's additional strong confirmation that has demonstrated the reliability and trustworthiness of decomposition method use Wazwaz [5]. The use of decomposition method by Wazwaz [7] was ac tually proved by giving the numerical outcomes that were of the fifth order boundary value problems and the use of the sixth–degree B-spline method showed also its efficiency in making a noticeable attainment in displaying the differences and similarities (contrast) between the errors. According to the numerical outcomes given by Wazwaz for the purpose of demonstrating the use of decomposition method, and the use of sixth–degree B-spline method, it becomes so clear those numerical outcomes indicate that decomposition method was more accurate and easy than B-spline method. This study re veals a real possibility of using new modification of the (MADM) which is suggested and offered in this study as a validate, and reliable modification by which the standard difficulties of (ADM) could be faced and tackled for solving of higher-order boundary value problems under various Kinds of dif ferent conditions to solve an equation at more than one condition. Gener ally, what can be briefly said about (MADM) as a final authenticate results of this study as a modified form of (ADM) that demonstrates its strength in giving a proven evidence for solving higher-order boundary value prob lems. The ideal way of successful application for MADM which also shows its meaningful, and accurate use by creating a canonical form which includes all boundary circumstances or conditions where extra calculations are not necessary to specify the zeroth component and other component obviously.

II. ANALYSIS OF THE METHOD

We provided the new differential operator L, for study the eq.(1),

$$L(.) = x^{-1}\frac{d^{m+1}}{dx^{m+1}}x^{m+2}\frac{d}{dx}x^{-m-1}\frac{d^{n-m}}{dx^{n-m}}(.). \tag{4}$$

We can write eq.(1) as

$$Ly = f(x,y,y',y'',...,y^{(n+1)}), \tag{5}$$

under one of the conditions (2) and (3), for two inverse operators L^{-1} are given, respectively as

$$L^{-1} = \underbrace{\int_0^x\int_0^x\int_0^x ...\int_0^x}_{(n-m)} x^{m+1}\int_b^x x^{-m-2}\underbrace{\int_0^x\int_0^x ...\int_0^x}_{(m+1)} x(.)\underbrace{dxdx...dx}_{(n+2)-times} \tag{6}$$

$$L^{-1} = \underbrace{\int_{a_0}^x\int_{a_1}^x\int_{a_2}^x ...\int_{a_{n-1}}^x}_{(n-m)} x^{m+1}\int_{a_n}^x x^{-m-2}\underbrace{\int_0^x\int_0^x ...\int_0^x}_{(m+1)} x(.)\underbrace{dxdx...dx}_{(n+2)-times} \tag{7}$$

Applying L^{-1} on both sides (5), we give

$$y(x) = \alpha(x) + L^{-1}f(x,y,y',...,y^{(n+1)}), \tag{8}$$

where $\alpha(x)$ which represent the term comes out from conditions. The Adomain decomposition method provided the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{9}$$

and the non-linear $f(x,y,y^0,y^{00},...,y^{(n+1)})$ by an infinite series of polynomials

$$f(x,y,y',y'',...,y^{(n+1)}) = \sum_{n=0}^{\infty} A_n, \tag{10}$$

where $y_n(x)$ of the solution $y(x)$ and A_n are Adomain polynomials [2]. By

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ..., \quad (11)$$

which gives

$$A_0 = N(y_0),$$
$$A_1 = y_1 N'(y_0),$$
$$A_2 = y_2 N'(y_0) + y_1^2 \frac{1}{2} N''(y_0),$$
$$A_3 = y_3 N'(y_0) + y_1 y_2 N''(y_0) + y_1^3 \frac{1}{3!} N'''(y_0)$$
$$\ldots \quad , \quad (12)$$

Substituting from eq.(9) and eq.(10) into eq.(8), we have

$$\sum_{n=0}^{\infty} y_n(x) = \alpha(x) + L^{-1}\sum_{n=0}^{\infty} A_n, \quad (13)$$

the components y_n can be specified as

$$y_0 = \alpha(x),$$
$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

Which gives

$$y_0 = \alpha(x),$$
$$y_1 = L^{-1}A_0,$$
$$y_2 = L^{-1}A_1,$$
$$y_3 = L^{-1}A_2, \quad (14)$$
$$\vdots$$

Addition the plan (14) with (12) can enable us to determine $y_n(x)$ and hence the series solution of $y(x)$ defined by (9) follows directly. For numerical use the $n-$term approximate

$$\phi_n = \sum_{k=0}^{n-1} y_k \quad , \quad (15)$$

can be used to approximate the exact solution. The approach above can be support by testing it on a variety of several linear and nonlinear BVP.

\III. NUMERICAL EXAMPLES

In this part, we will discussing for example, when n=1,2,5, in a differential operator (4). We apply the introduces algorithm on two third order nonlinear boundary value problems at m=0& m=1, two fourth order non-linear boundary value problems at m=0 & m=1 and one seventh order non-linear boundary value problem at m=0& m=1 and in every one case two boundary conditions.

3.1 Example

At n=1 and m=0, we give non-linear equation of third order:

$$y'''(x) = y^2 - y(e^x + x) + e^x, \quad (16)$$

with one of the following condition:

$$y(0) = 1, y'(0) = 2, y'(\frac{1}{2}) = 2.65,$$
$$y(1) = 3.72, y'(\frac{1}{2}) = 2.65, y'(0) = 2,$$

The exact solution is $y(x) = e^x + x$.

Re-written eq.(16), as

$$Ly = y^2 - y(e^x + x) + e^x, \quad (17)$$

of an operator (4), give

$$L(.) = x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d}{dx}(.),$$

of two inverse operators, under one of the following condition, respectively

$$L^{-1}(.) = \int_0^x x\int_{\frac{1}{2}}^x x^{-2}\int_0^x x(.)\ dxdxdx,$$

$$L^{-1}(.) = \int_1^x x\int_{\frac{1}{2}}^x x^{-2}\int_0^x x(.)\ dxdxdx.$$

Applying L^{-1} on both sides of (17), respectively we get

$y(x) = 1 + 2x + 0.65x2 + ex + L-1y2 - L-1y(ex + x),$

$y(x) = 1.07 + 2x + 0.65x2 + ex + L-1y2 - L-1y(ex + x),$

using ADM for $y^2(x)$, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 1 + 2x + 0.65x^2 + e^x + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n(e^x + x), n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 1.07 + 2x + 0.65x^2 + e^x + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n(e^x + x), n \geq 0,$$

we obtain the relative formal

$y_0 = 1 + 2x + 0.491279x^2 + 0.166667x^3 + ... + 2.7557310^{-7}x^{10},$

$y_0 = 1.00044+2x+0.501279x^2+0.166667x^3+0.0416667x^4+...+2.7557310^{-7}x^{10},$

$yn+1 = L-1An - L-1yn(ex + x), n \geq 0,$

employing Adomain polynomial A_n, for y^2, when n=0,1,2, gives

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

the first few components are as follows, respectively

$$y_1 = 0.0000752066\,x^2 - 2.77556\,10^{-17}\,x^3 - 1.38778\,10^{-16}\,x^4 - 0.000145355\,x^5$$
$$+ ... + 1.00941\,10^{-7}\,x^{10},$$

$$y_2 = -5.78632\,10^{-7}\,x^2 + 1.25344\,10^{-6}\,x^5 + 1.25344\,10^{-6}\,x^6 + 1.72817\,10^{-7}\,x^7$$
$$+ ... + 5.28649\,10^{-7}\,x^{10},$$

$$y_1 = -0.0000751438 - 0.0000855731\,x^2 + 0.0000732726\,x^3 +$$
$$0.0000366202\,x^4 + ... + 1.49212\,10^{-8}\,x^{10},$$

$$y_2 = 7.1956\,10^{-6} + 0.0000132446\,x^2 - 0.000012535\,x^3 - 6.26198\,10^{-6}\,x^4$$
$$+ ... + 9.44112\,10^{-8}\,x^{10},$$

the solution in a series from are given by

$$y(x) = y_0 + y_1 + y_2 = 1 + 2\,x + 0.491353\,x^2 + 0.166667\,x^3 + 0.0416667\,x^4 + 0.00818923\,x^5$$
$$+ 0.00124479\,x^6 + 0.000178183\,x^7 + 0.0000200803\,x^8 + 1.17574\,10^{-6}\,x^9$$
$$+ ... + 3.59999\,10^{-18}\,x^{25},$$

$$y(x) = y_0 + y_1 + y_2 = 1.00037 + 2\,x + 0.501206\,x^2 + 0.166727\,x^3$$
$$+ 0.041697\,x^4 + 0.00140989\,x^6 + 0.000202205\,x^7 + 0.0000258045\,x^8 + 3.05884\,10^{-6}\,x^9$$
$$+ ... + 3.83708\,10^{-18}\,x^{25},$$

Table 1. The comparison of Exact solution $y(x) = e^x + x$, and Approximate solution

x	Exact solution	MADN at the first condition	Absolute Error	MADN at the second condition	Absolute Error
0.0	1.00000	1.00000	0.00000	1.00037	0.00037
0.1	1.20517	1.20508	0.00009	1.20555	0.00037
0.2	1.42140	1.42106	0.00034	1.42182	0.00042
0.3	1.64986	1.64908	0.00078	1.65034	0.00048
0.4	1.89182	1.89044	0.00138	1.89239	0.00057
0.5	2.14872	2.14655	0.00217	2.14940	0.00068
0.6	2.42212	2.41899	0.00313	2.42294	0.00082
0.7	2.71375	2.70947	0.00428	2.71475	0.00100
0.8	3.02554	3.01992	0.00562	3.02674	0.00120
0.9	3.35960	3.35243	0.00717	3.36104	0.00144
1.0	3.71828	3.70932	0.00896	3.72000	0.00172

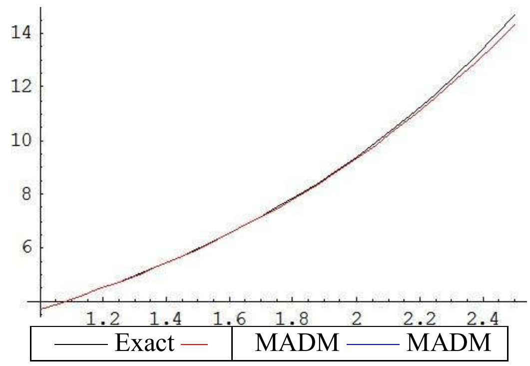

Figure 1: Showing the representation of the approximate solutions which is very close to the exact solution $y(x) = e^x + x$,

We note from the graphical and table above shows approximate solutions that is a lot very close to exact solution. Therefore the method is very quickly to get exact solution.

3.2 Example

In the same case, we give example for non-linear equation of third order:

$$y''' = y^3 - x^6, \tag{18}$$

under one of the following condition:

$$y(0) = 0, y'(0) = 0, y(\frac{1}{2}) = \frac{1}{4}$$

$$y(1) = 1, y(0) = 0, y'(0) = 0.,$$

The exact solution is $y(x) = x^2$. From an operator (4), when m=1,n=1, we have

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.)$$

Re-written eq.(18), as

$$Ly = y^3 - x^6, \tag{19}$$

of two inverse operators, respectively

$$L^{-1}(.) = x^2\int_{\frac{1}{2}}^{x} x^{-3}\int_0^x\int_0^x x(.)\,dxdxdx,$$

$$L^{-1}(.) = x^2\int_{1}^{x} x^{-3}\int_0^x\int_0^x x(.)\,dxdxdx.$$

Applying L^{-1} on both sides(19), respectively we give

$$y(x) = x2 - L-1x6 + L-1y3,$$

$$y(x) = x2 - L-1x6 + L-1y3,$$

using ADM for $y^3(x)$, as yield ,

$$\sum_{n=0}^{\infty} y_n(x) = x^2 - L^{-1}x^6 + L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1.00002x^2 - 0.00198413x^9,$$

$$y_0 = 1.00198x^2 - 0.00198413x^9,$$
$$yn{+}1 = L{-}1An,n \geq 0,$$

employing Adomain polynomial A_n, for y^3, we have the first few components as follows, respectively

$$y_1 = 1.00002x^2 - 0.00198413x^9,$$
$$y = -0.0000155016x^2 + 0.00198422x^9$$
$$-1.771610^{-6}x^{16} + 1.1114710^{-9}x^{23} - 3.206510^{-13}x^{30},$$
$$y_3 = -4.0410410^{-14}x^2 + 6.5082310^{-12}x^9 + 2.7235710^{-9}x^{23}$$
$$+ ... + 2.2519510^{-18}x^{44},$$
$$y_1 = 0.0000101238x^2 - 0.0000119173x^9 + 1.7962710^{-6}x^{16}$$
$$+ ... + 2.1364910^{-19}x^{44},$$
$$y_2 = 0.0000101238x^2 - 0.0000119173x^9 - 2.7469510^{-9}x^{23}$$
$$+ ... + 2.1364910^{-19}x^{44},$$
$$y_3 = 1.x^2 + 2.4375710^{-8}x^9 - 7.2889610^{-9}x^{16} + ...2.056610^{-18}x^{44},$$

The first terms, the approximate is following, respectively

$$y(x) = y_0{+}y_1{+}y_2{+}y_3 = x^2{+}1.4306910^{-12}x^9{-}5.493710^{-11}x^{16}{+}1.111810^{-9}x^{23}$$
$$+ ... + 2.0387210^{-18}x^{44},$$
$$y(x) = y = y_0{+}y_1{+}y_2{+}y_3 = 1.x^2{+}2.4375710^{-8}x^9{-}7.2889610^{-9}x^{16}{+}1.1561410^{-9}x^{23}$$
$$+ ... + 2.056610^{-18}x^{44},$$

Table 2. We formulated the exact solution with the approximate MADM in [0.1,1]

x	Exact solution	MADN at the first condition	Absolute Error	MADN at the second condition	Absolute Error
0.1	0.01	0.01	0.00	0.01	0.00
0.2	0.04	0.04	0.00	0.04	0.00
0.3	0.09	0.09	0.00	0.09	0.00
0.4	0.16	0.16	0.00	0.16	0.00
0.5	0.25	0.25	0.00	0.25	0.00
0.6	0.36	0.36	0.00	0.36	0.00
0.7	0.49	0.49	0.00	0.49	0.00
0.8	0.64	0.64	0.00	0.64	0.00
0.9	0.81	0.81	0.00	0.81	0.00
1.0	1.00	1.00	0.00	1.00	0.00

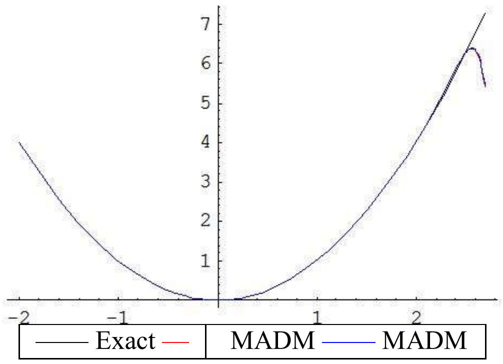

Figure 2: Showing the representation of the approximate solutions which is very close to the exact solution $y(x) = x^2$. Obviously, the former example, we have the exact solution. Thus the good method and its effectiveness.

3.3 Example

At n=2& m=0, we study non-linear equation of fourth order:

$$y'''' = (y')^2 - yy'', \tag{20}$$

with one of the following condition:

$$y(0) = 0, y'(0) = 1, y''(0) = 1, y''(\frac{1}{2}) = 1.65,$$
$$y(\frac{1}{2}) = 0.6787, y'(\frac{1}{3}) = 1.3956, y''(1) = 2.7183, y''(0) = 1.$$

The exact solution is $y(x) = e^x - 1$.

From an operator(4), we get

$$L(.) = x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d^2}{dx^2}(.).$$

Re-written eq.(20), as

$$Ly = (y')^2 - yy'', \tag{21}$$

of two inverse operators, respectively

$$L^{-1}(.) = \int_0^x \int_0^x x \int_{\frac{1}{2}}^x x^{-2} \int_0^x x(.)\ dxdxdxdx,$$
$$L^{-1}(.) = \int_{\frac{1}{2}}^x \int_{\frac{1}{3}}^x x \int_1^x x^{-2} \int_0^x x(.)\ dxdxdxdx.$$

Applying L^{-1} on both sides (21), we get

$y(x) = x + 0.5x^2 + 0.22x^3 + L^{-1}(y')^2 - L^{-1}yy^{00}$, $y(x) = 0.045 + 0.9668x + 0.5x^2 + 0.2864x^3 + L^{-1}(y^0)^2 - L^{-1}yy^{00}$,

employing ADM for $y^{02}(x)$, as yield

$$\sum_{n=0}^{\infty} y_n(x) = x + 0.5x^2 + 0.22x^3 + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n y_n'', n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 0.045 + 0.9668x + 0.5x^2 + 0.2864x^3 + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n y_n'', n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = x + 0.5x^2 + 0.22x^3,$$
$$y_0 = 0.045 + 0.9668x + 0.5x^2 + 0.2864x^3,$$
$$y_{n+1} = L^{-1}An - L^{-1}y_n y''_n, n \geq 0,$$

applying Adomain polynomial A_n, for the non-linear term $(y^0)^2$, when for n=0,1,2, gives

$$A_0 = y_0^2,$$
$$A_1 = 2y_0 y_1,$$
$$A_2 = y_1^2 + 2y_0 y_2,$$

we get,

$$y_1 = -L^{-1}y_0 y_0'' + L^{-1}(y_0')^2,$$
$$y_2 = -L^{-1}y_1 y_1'' + L^{-1}(2y_0' y_1'),$$

the first few components as follows, respectively

$$y_1 = -0.0497335\,x^3 + 0.0416667\,x^4 + 0.00833333\,x^5 + ... + 0.0000864286\,x^8,$$
$$y_2 = 0.0000645216\,x^3 - 0.000143682\,x^7 + ... + 0.0000232251\,x^{10},$$
$$y_3 = -8.85146\,10^{-7}\,x^3 + 1.07536\,10^{-6}\,x^6 + ... + 3.23442\,10^{-7}\,x^{10},$$

$$y_1 = -0.00444651 + 0.0328041\,x - 1.04075\,10^{-17}\,x^2 + ... + 0.000146473\,x^8,$$
$$y_2 = -4.20624\,10^{-6} + 0.0000323662\,x - 0.000112691\,x^3 + ... + 3.31524\,10^{-7}\,x^{10},$$

The first terms, the approximate solution is following

$$y(x) = y_0 + y_1 + y_2 + y_3 = x + 0.5\,x^2 + 0.170331\,x^3 + 0.0416667\,x^4 + 0.00833333\,x^5 + 0.00138889\,x^6 + 0.000380127\,x^7 + 0.0000641907\,x^8 + 9.26112\,10^{-6}\,x^9 + ... + 1.92722\,10^{-12}\,x^{19},$$

$$y(x) = y_0 + y_1 + y_2 = 0.000049288 + 0.999636\,x + 0.5\,x^2 + 0.169045\,x^3 + 0.0388033\,x^4 + 0.00796616\,x^5 + 0.00139463\,x^6 + 0.000676696\,x^7 + 0.000169569\,x^8 - 9.24522\,10^{-6}\,x^9 + 3.31524\,10^{-7}\,x^{10} + 2.88628\,10^{-7}\,x^{11} + ... + 2.33708\,10^{-12}\,x^{18},$$

Table 3. Comparison between Exact solution $y(x) = e^x - 1$, and MADM

x	Exact solution	MADN at the first condition	Absolute Error	MADN at the second condition	Absolute Error
0.1	0.105171	0.105171	0.000000	0.105186	0.000015
0.2	0.221403	0.221432	0.000029	0.221394	0.000009
0.3	0.349859	0.349958	0.000099	0.349839	0.000020
0.4	0.491825	0.492060	0.000235	0.491805	0.000020
0.5	0.648721	0.649181	0.000046	0.648700	0.000021
0.6	0.822119	0.822916	0.000797	0.822080	0.000039
0.7	1.013750	1.015030	0.001280	1.013660	0.000090
0.8	1.225540	1.227470	0.001930	1.225350	0.000190
0.9	1.459600	1.462390	0.002790	1.459250	0.000350
1.0	1.718280	1.722200	0.003920	1.717730	0.000550

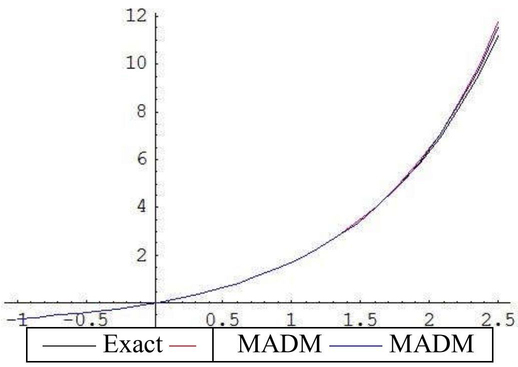

Figure 3: Comparison between Exact solution and MADM

We see of tables and fingers above that clearly the MADM is precise, more dynamic and converges to the exact solution.

3.4 Example

When n=2, we give example for non-linear of fourth order:

$$y^{(4)} = e^x y^2, \tag{22}$$

under one of the following condition:

$$y(0) = 1, y'(0) = -1, y''(0) = 1, y'(\frac{1}{2}) = -0.61,$$

$$y(\frac{1}{2}) = 0.61, y'(1) = -0.37, y'(0) = -1, y''(0) = 1.$$

The exact solution is $y(x) = e^{-x}$. From an operator (4), m=1, we get

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.)\frac{d}{dx}.$$

Re-written eq.(22), as

$$Ly = e^x y^2, \tag{23}$$

of two inverse operators, respectively

$$L^{-1}(.) = \int_0^x x^2 \int_{\frac{1}{2}}^x x^{-3} \int_0^x \int_0^x x(.)\ dxdxdxdx,$$

$$L^{-1}(.) = \int_{\frac{1}{2}}^x x^2 \int_1^x x^{-3} \int_0^x \int_0^x x(.)\ dxdxdxdx.$$

Applying L^{-1} on both sides(23), we give

$$y(x) = 1 - x + 0.5x^2 - 0.213x^3 + L^{-1}e^x y^2,\ y(x) = 1.0004 - x + 0.5x^2 - 0.123x^3 + L^{-1}e^x y^2,$$

employing ADM for $y^{02}(x)$, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 1 - x + 0.5x^2 - 0.123x^3 + L^{-1}\sum_{n=0}^{\infty} e^x A_n, n \geq 0 ,$$

$$\sum_{n=0}^{\infty} y_n(x) = 1.0004 - x + 0.5x^2 - 0.123x^3 + L^{-1}\sum_{n=0}^{\infty} e^x A_n, n \geq 0 ,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 - x + 0.5x^2 - 0.123x^3,$$
$$y_0 = 1.0004 - x + 0.5x^2 - 0.123x^3, \ y$$
$$n+1 = L-1exAn, n \geq 0,$$

the first few components as follows, respectively

$$y_1 = -0.0246058x^3 + 0.0416667x^4 - 0.00833333x^5 + \ldots + 1.5037510^{-7}x^{10},$$
$$y_2 = 4.6456610^{-6}x^3 - 0.0000585853x^7 + 0.0000496032x^8 + \ldots + 1.0035610^{-6}x^{10},$$
$$y_1 = 0.00315959 - 2.2144610^{-18}x - 4.7057310^{-18}x^2 + \ldots + 1.0170610^{-7}x^{10},$$
$$y_2 = 1.00001 + x + 0.5x^2 + 0.166444x^3 + 0.0419301x^4 + \ldots + 5.3390110^{-8}x^{10},$$

The first terms, the approximate solution is following

$$y(x) = y_0 + y_1 + y_2 = 1 - x + 0.5x^2 - 0.237601x^3 + 0.0416667x^4 + \ldots + 1.2739710^{-6}$$
$$x^{10}, \ y(x) = y_0 + y_1 + y_2 = 1.00357 - x + 0.5x^2 - 0.167435x^3 + 0.0419648x^4 + \ldots + 5.1325410^{-10}x^{14},$$

Table 4. Comparison between Exact solution $y(x) = e^{-x}$, and MADM

x	Exact solution	MADN at the first condition	Absolute Error	MADN at the second condition	Absolute Error
0.1	0.904837	0.904766	0.00007	0.908408	0.00357
0.2	0.818731	0.818163	0.00057	0.822297	0.00367
0.3	0.740818	0.738903	0.00192	0.744372	0.00355
0.4	0.670320	0.665780	0.00454	0.673850	0.00353
0.5	0.606531	0.597663	0.00887	0.610025	0.00349
0.6	0.548812	0.533485	0.01533	0.552256	0.00344
0.7	0.496585	0.472241	0.02434	0.499965	0.00338
0.8	0.449329	0.412975	0.03635	0.452629	0.00330
0.9	0.406570	0.354780	0.05180	0.409777	0.00321
1.0	0.367879	0.296778	0.07110	0.370980	0.00310

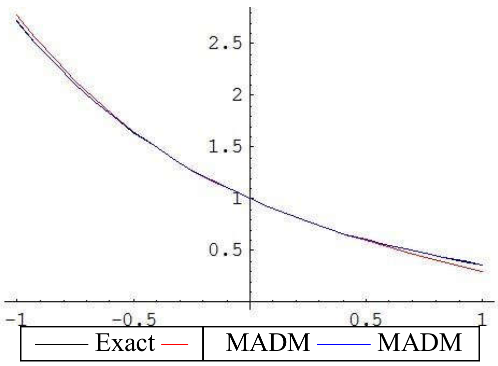

Figure 4: Comparing between Exact solution and MADM

In the same way, we got the results of the exact solution, its excellentmethod.

3.5 Example

This example from seventh order, we show tow cases for a differential operator (4), at m=0 and m=1, with one conditions (2) or (3). The first case,

$$y^{(7)} = (1 + x)^3 - y^3, \qquad (24)$$

with one of the following condition:

$$y(0)=1, y'(0)=1, y''(0)=0, y'''(0)=0, y''''(0)=0, y^{(5)}(0)=0, y^{(5)}(1)=0,$$

$$y(1)=2, y'(\frac{1}{2})=1, y''(\frac{1}{3})=0, y'''(\frac{1}{4})=0, y^{(4)}(\frac{1}{5})=0, y^{(5)}(\frac{1}{7})=0, y^{(5)}(0)=0.$$

With the exact solution $y(x) = x + 1$.

From an operator(4) at m=0 and n=5, we have

$$L(.) = x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d^5}{dx^5}(.).$$

Re-written eq.(24), as

$$Ly = (1+x)^3 - y^3, \quad (25)$$

of two inverse operators, respectively

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x x \int_1^x x^{-2} \int_0^x x(.)\, dxdxdxdxdxdxdx,$$

$$L^{-1}(.) = \int_1^x \int_{\frac{1}{2}}^x \int_{\frac{1}{3}}^x \int_{\frac{1}{4}}^x \int_{\frac{1}{5}}^x x \int_{\frac{1}{7}}^x x^{-2} \int_0^x x(.)\, dxdxdxdxdxdxdx.$$

Applying L^{-1} on both sides (24), we give respectively

$$y(x) = 1 + x + L^{-1}(1+x)^3 - L^{-1}y^3,$$

employing ADM for $y^3(x)$, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x + L^{-1}(1+x)^3 - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + L^{-1}(1+x)^3,$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0,$$

the first few components as follows, respectively

$$y_0 = 1 + x - 0.00180556\, x^6 + 0.000198413\, x^7$$

$$+0.0000744048\, x^8 + 0.0000165344\, x^9 + 1.65344\, 10^{-6}\, x^{10},$$

$$y_1 = 0.00180518\, x^6 - 0.000198413\, x^7 - 0.0000744048\, x^8 + ... + 1.95306\, 10^{-14}\, x^{20},$$

$$y_2 = 3.75522\, 10^{-7}\, x^6 - 6.26172\, 10^{-10}\, x^{13} + ... + 3.90471\, 10^{-14}\, x^{20},$$

$$y_0 = 0.999827 + 0.999994\, x - 1.95044\, 10^{-6}\, x^2 + ... + 1.65344\, 10^{-6}\, x^{10},$$

$$y_1 = 0.00017307 + 5.69695\, 10^{-6}\, x + 1.94954\, 10^{-6}\, x^2 + ... + 8.7114\, 10^{-29}\, x^{37},$$

$$y_2 = 7.42729\, 10^{-8} + ... + 3.88892\, 10^{-38}\, x^{50},$$

The first terms, the approximate solution is following

$$y(x) = y_0 + y_1 + y_2 = 1 + x - 0.0018048\, x^6 + 0.000198413\, x^7 + ... + 9.85861\, 10^{-38}\, x^{50},$$

$$y(x) = y_0 + y_1 + y_2 = 1. + 1.\, x - 5.20394\, 10^{-13}\, x^2 + 8.71935\, 10^{-15}\, x^3 + ... + 3.88892\, 10^{-38}\, x^{50},$$

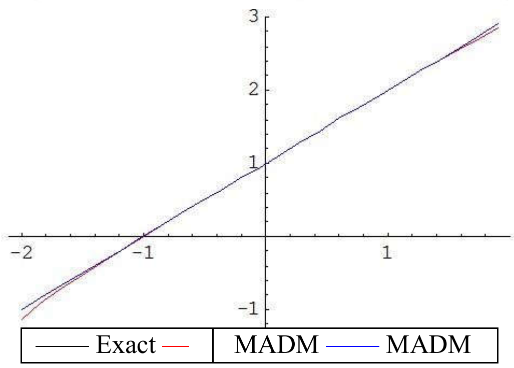

Figure 5.1 Comparing between Exact solution and MADM

We will study the same example at m=1, with one of the following conditions and difference differential operator

$$y(0)=1, y'(0)=1, y''(0)=0, y'''(0)=0, y''''(0)=0, y^{(5)}(0)=0, y^{(4)}(1)=0,$$

$$y(\frac{1}{2})=\frac{3}{2}, y'(\frac{2}{3})=1, y''(\frac{1}{4})=0, y'''(\frac{1}{5})=0, y^{(4)}(1)=0, y^{(4)}(0)=0, y^{(5)}(0)=0.$$

From an operator (4), we get

$$L(.)=x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.)\frac{d^4}{dx^4},$$

of two inverse operators, respectively

$$L^{-1}(.)=\int_0^x\int_0^x\int_0^x\int_0^x x^2\int_1^x x^{-3}\int_0^x\int_0^x x(.)\,dxdxdxdxdxdxdx,$$

$$L^{-1}(.)=\int_{\frac{1}{2}}^x\int_{\frac{2}{3}}^x\int_1^4\int_{\frac{1}{5}}^x x^2\int_1^x x^{-3}\int_0^x\int_0^x x(.)\,dxdxdxdxdxdxdx,$$

and the same way, we get the first few components as follows

$$y_0=1+x-0.000972222\,x^6+0.000198413\,x^7+...+1.65344\,10^{-6}\,x^{10},$$

$$y_1=0.000972185\,x^6-0.000198413\,x^7-0.0000744048\,x^8+...+8.7114\,10^{-29}\,x^{37},$$

$$y_2=7.5857\,10^{-9}\,x^6-3.37227\,10^{-10}\,x^{13}+...+4.113\,10^{-38}\,x^{50},$$

$$y_0=0.999765+1.00049\,x-0.0000549137\,x^2+0.00014302\,x^3+...+1.65344\,10^{-6}\,x^{10},$$

$$y_1=0.000234648-0.000487628\,x+0.0000549036\,x^2+...+4.12376\,10^{-18}\,x^{25},$$

$$y_2=3.76026\,10^{-8}-7.87435\,10^{-8}\,x+1.1797\,10^{-8}\,x^2+...+5.45732\,10^{-17}\,x^{25},$$

The first terms of the approximate solution is following

$$y(x)=y_0+y_1+y_2=1+x-4.91636\,10^{-12}\,x^6+...+4.113\,10^{-38}\,x^{50},$$

$$y(x)=y_0+y_1+y_2=1.+1.\,x+1.68813\,10^{-9}\,x^2-4.38757\,10^{-9}\,x^3+...+5.60133\,10^{-36}\,x^{50},$$

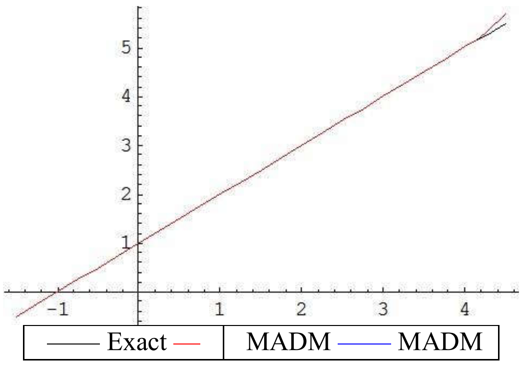

Figure 5.2 Comparing between Exact solution and MADM

The approximate solutions of the Figure 5.1 and the Figure 5.2 which is very close to the exact solution. Therefor the method is very effective.

IV. CONCLUSION

This is method beneficial, active and we can get it in simple ways. It is noticed that when we use this method, we can get an accurate results and sometimes the exact solution. We have also found out that this method has a real efficiency and it can be developed and used to find out the solutions the differential operator of the inverse operator by boundary conditions in general. It is also noticed from those illustrative examples, we can get an approximate solution by using an illustrative Tables and Figures.

ACKNOWLEDGMENT

First of all, all praise is to Allah for all the blessings. He endows me with. In my research journey, I am blessed to be under the supervision of Professor Yahya Qiad Hassan. He has been so informative and cooperative giving me guidelines to make my work a piece of success. I also would like to thank my family for everything they do for me. Special thanks to My parents Baa Ali and Maa Akhlasmy for his help and support, for being so kind and for standing by my side whenever I need him. My husband, Nabeel , the source of hope, love and inspirations, I thank you so much for everything. I thank you for your constant prayers and for encouraging me to continue. I cannot forget my children who were so patient to endure life away from me. I am grateful to everyone who helped me in my research in any way. Thank you all.

REFERENCES

[1]. A.Boutayeb, and A.H.Twizell,"Finite-difference methods for the solution of special eight-order boundary value problems," Int. J. Com. Math., vol. 48, pp. 63-75, 1993.

[2]. A.M.Wazwaz,"A new algorithm for calculating Adomian polynomials for nonlinear operators," Appl. Math. Comput., vol. 111, pp. 33-51, 2000.

[3]. A.M.Wazwaz,"A new algorithm for solving boundary for higher-order integral differential equations," Appl. Math. Comp., vol. 118, pp. 327-342, 2001.

[4]. A.M.Wazwaz,"A reliable algorithm for obtaining positive solutions for nonlinear boundary-value problems," Comput. Math. Appl., vol. 41, pp.1237-1246, 2001.

[5]. A.M.Wazwaz,"Blow-up for solution of some linear wave equations with mixed nonlinear boundary conditions," Appl. Math. Comp., vol. 123, pp.133-140, 2001.

[6]. A.M.Wazwaz,"The modified Adomian decomposition method for solving linear and nonlinear boundary-value problems of tenth-order and 12thorder," Int. J. Nonlinear Sci. Numr. Simul, vol. 1, pp. 17-24, 2000.

[7]. A.M.Wazwaz,"The numerical solution of fifth-order BVP by the decomposition method," J. Comput. Appl. Math., vol. 136, pp.259-270, 2001.

[8]. E.Deeba, S.Khuri and S.Xie,"An algorithm for solving boundary value problems," J. Comp. Phy. vol. 159, pp. 125-138, 2000.

[9]. G.Adomian, and R.Rach,"Analytic solution of nonlinear boundary value problems in severl dimensions," Math. Anal. Appl. vol. 174, pp. 118-127, 1993.

[10]. G.Adomian,"Nonlinear stochastic operator equations," Acadmic Press, SanDiego, CA, 1986.

[11]. G.Adomian,"Solving frontier problems of physics: the decomposition method," Kluwer academic publishers London, 1994.

[12]. KNS.Viswanadham, and Y.Showri Raju,"Quintic B-spline Collocation Method for Eighth Order Boundary Value Problems," Advances in Computational Mathematics and its Applications, vol. 1, pp. 47-52, 2012.

[13]. R.P.Agarwal,"Boundary value problems for high-order differential equations," World Scientific, Singapore, 1986.

[14]. S.Chandrasekhar,"Hydrodynamic and Hydromagnetic Stability,"Dover, New york,NY, 1981.

[15]. Y.Q.Hasan,"Modified Adomian decomposition method for second order singular initial value problems," Advances in Computational Mathematics and its Applications., vol. 1, pp. 94-99, 2012.