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WEAK FORMS OF ω−OPEN SETS

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Abstract:-
The principle purpose of this paper is to introduce and study some new classes of sets in topological spaces which are finer than the classes of open sets and ω−open sets. The continuity via these classes will be introduced and studied.

Keywords:-
Open set; Generalized Open set; Decomposition of continuity.

AMS classification: Primary 54A05, 54A10, 54C10

1 INTRODUCTION

In general topology, many authors introduced and studied some classes of weak or strong forms of open sets in topological spaces. In 1970 Levine, [6], introduced the notion of a gen eralized open sets which is weak form of open sets. In 1982 Hdeib [4] introduced the notion of a $\omega-$open sets which is weak form of open sets. In 1983 the authors [1] introduced the weak form for an open set which is called a $\beta-$open set. In 2005 Al-Zoubi [2] introduced the generalization property of $\omega-$open sets to get the weak form of $\omega-$open sets. In 2009 Noiri and Noorani [7] introduced the notion of $\beta\omega-$open set which is weak form for a $\omega-$open sets and a $\beta-$open sets.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we introduce the concept of generalized $\beta\omega-$open sets by utilizing the $\beta\omega-$closure operator. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the notions of $\beta\omega-$continuous, generalized $\beta\omega-$continuous, Slightly and Contra $\beta\omega-$Continuous functions.

2 Preliminaries

For a topological space (X,τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

Theorem 2.1. [5] For a topological space (X,τ) and $A,B \subseteq X$, if B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

Theorem 2.2. [5] For a topological space (X,τ),

1. $Cl(X - A) = X - Int(A)$ for all $A \subseteq X$.
2. $Int(X - A) = X - Cl(A)$ for all $A \subseteq X$.

Definition 2.3. [6] A subset A of a topological space (X,τ) is called *generalized closed* (simply $g-$closed) *set*, if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X,τ). The complement of $g-$closed set is called *generalized open* (simply $g-$open) *set*.

Theorem 2.4. [6] Every closed set is a $g-$closed set.

Definition 2.5. A topological space (X,τ) is called:

1. $T_{1/2}-$*space* [6] if every $g-$closed set is closed set.
2. T_1-*space* [5] if for each disjoint point $x \neq y \in X$, there are two open sets G and H in X such that $x \in H$, $y \in G$, $x \notin G$ and $y \notin H$.

Theorem 2.6. [3] A topological space (X,τ) is $T_{1/2}-$*space* if and only if every singleton set is open or closed set.

Theorem 2.7. [5] A topological space (X,τ) is T_1-space if and only if every singleton set is closed set.

Definition 2.8. [4] A subset A of a space X is called $\omega-$*open set* if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set. The set of all $\omega-$closed sets in X denoted by $\omega C(X,\tau)$ and the set of all $\omega-$open sets in X denoted by $\omega O(X,\tau)$.

Theorem 2.9. [4] Every open set is $\omega-$open set.

Theorem 2.10. [4] For a topological space (X,τ), the pair $[X,\omega O(X,\tau)]$ forms a topological space.

For a topological space (X,τ) and $A \subseteq X$, the $\omega-$closure set of A is defined as the intersection of all $\omega-$closed subsets of X containing A and is denoted by $Cl_\omega(A)$. The $\omega-$interior set of A is defined as the union of all $\omega-$open subsets of X contained in A and is denoted by $Int_\omega(A)$.

Definition 2.11. [2] A subset A of a space X is called *generalized $\omega-$closed set* (simply $g\omega-$closed) *set* if $Cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of generalized $\omega-$closed set is called generalized $\omega-$open set (simply $g\omega-$open) *set*.

Theorem 2.12. [2] Every $g-$closed set is a $g\omega-$closed set.

Definition 2.13. [7] A subset A of a topological space (X,τ) is called $\beta\omega-$open set if $A \subseteq Cl(Int_\omega(Cl(A)))$. The complement of $\beta\omega-$open set is called $\beta\omega-$closed set. The set of all $\beta\omega-$closed sets in X denoted by $\beta\omega C(X,\tau)$ and the set of all $\beta\omega-$open sets in X denoted by $\beta\omega O(X,\tau)$.

Theorem 2.14. [7] The union of arbitrary of $\beta\omega-$open sets is $\beta\omega-$open set.

Theorem 2.15. [7] Every $\omega-$open set is $\beta\omega-$open set.

Definition 2.16. [5] A function $f : (X,\tau) \to (Y,\rho)$ of a space (X,τ) into a space (Y,ρ) is called *continuous function* if $f^{-1}(U)$ is an open set in X for every open set U in Y.

Definition 2.17. A function $f : (X,\tau) \to (Y,\rho)$ of a space (X,τ) into a space (Y,ρ) is called:

1. $g-$*continuous function* [6] if $f^{-1}(U)$ is a $g-$open set in X for every open set U in Y.

2. *ω−continuous function* [4] if for each $x \in X$ and for an open set G in Y containing $f(x)$, there is a ω−open set U in X containing x such that $f(U) \subseteq G$.
3. *gω−continuous function* [2] if $f^{-1}(U)$ is a gω−open set in X for every open set U in Y.

Theorem 2.18. [6] Every continuous function is g−continuous function.

Theorem 2.19. [4] A function $f : (X,\tau) \rightarrow (Y,\rho)$ is a ω−continuous function if and only if $f^{-1}(U)$ is a ω−open set in X for every open set U in Y.

Theorem 2.20. [4] Every continuous function is ω−continuous function.

Theorem 2.21. [2] Every ω−continuous function is gω−continuous function.

Theorem 2.22. [2] Every g−continuous function is gω−continuous function.

3 Generalized βω−open sets

For a topological space (X,τ) and $A \subseteq X$, the βω−closure set of A is defined as the intersec tion of all βω−closed subsets of X containing A and is denoted by $Cl_{\beta\omega}(A)$. The βω−interior set of A is defined as the union of all βω−open subsets of X contained in A and is denoted by $Int_{\beta\omega}(A)$. From Theorem (2.14), $Cl_{\beta\omega}(A)$ is a βω−closed subsets of X and $Int_{\beta\omega}(A)$ is βω−open subsets of X.

Definition 3.1. A subset A of a topological space (X,τ) is called *generalized βω−closed* (simply $G_{\beta\omega}$−closed) *set*, if $Cl_{\beta\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X,τ). The complement of $G_{\beta\omega}$−closed set is called *generalized βω−open* (simply $G_{\beta\omega}$−open) *set*.
For a topological space (X,τ), the set of all $G_{\beta\omega}$−closed sets in X denoted by $G_{\beta\omega}C(X,\tau)$ and the set of all $G_{\beta\omega}$−open sets in X denoted by $G_{\beta\omega}O(X,\tau)$.

Example 3.2. For any topological space (X,τ), if X is a countable then it's clear that every subset of X is i a both $G_{\beta\omega}$−closed and $G_{\beta\omega}$−open set. That is,

$$G_{\beta\omega}O(X,\tau) = G_{\beta\omega}C(X,\tau) = P(X),$$

where $P(X)$ is the power of X.

Example 3.3. Let (R,τ_u) be the real usual topological space on the set of real numbers R.
The rational set Q is a $G_{\beta\omega}$−closed set, since the irrational set IR is a βω−open set, that is, $Cl_{\beta\omega}(Q) = Q$.

Theorem 3.4. Any a countable subset of a topological space (X,τ) is a $G_{\beta\omega}$−closed set in X.
Proof. Let A be a countable subset of a topological space (X,τ). Then A is a βω−closed set, that is, $Cl_{\beta\omega}(A) = A$. That is, A is a $G_{\beta\omega}$−closed set. □

Theorem 3.5. Every βω−open set is $G_{\beta\omega}$−open set.
Proof. Let A be βω−open subset of a topological space (X,τ). Then $X - A$ is βω−closed set. Hence $X - A = Cl_{\beta\omega}(X - A) \subseteq U$ whenever $X - A \subseteq U$ and U is open set. That is, A is $G_{\beta\omega}$−open set. □

Corollary 3.6. Every βω−closed set is $G_{\beta\omega}$−closed set.
The converse of the last theorem need not be true.

Example 3.7. In topological space (R,τ), R is the set of real numbers and $\tau = \{\emptyset, R, R - \{2,3\}\}$, the set $R - \{2\}$ is $G_{\beta\omega}$−closed set but it is not βω−closed set.

Theorem 3.8. Let (X,τ) be a topological space. If (X,τ) is a $T_{1/2}$−space then every $G_{\beta\omega}$−closed set in X is βω−closed set in X.
Proof. Let A be a $G_{\beta\omega}$−closed set in X. Suppose that A is not βω−closed set. Then there is at least $x \in Cl_{\beta\omega}(A)$ such that x /∈ A. Since (X,τ) is a $T_{1/2}$−space then by Theorem (2.6), $\{x\}$ is an open or closed set in X. If $\{x\}$ is a closed set in X then $X - \{x\}$ is an open. Since x /∈ A then $A \subseteq X - \{x\}$. Since A is a $G_{\beta\omega}$−closed set and $X - \{x\}$ is an open subset of X containing A, then $Cl_{\beta\omega}(A) \subseteq X - \{x\}$. Hence $x \in X - Cl_{\beta\omega}(A)$ and this a contradiction, since $x \in Cl_{\beta\omega}(A)$. If $\{x\}$ is an open set then it is βω−open set. Since $x \in Cl_{\beta\omega}(A)$ then we have $\{x\} \cap A$ 6= ∅. That is, $x \in A$ and this a contradiction. Hence A is a βω−closed set in X. □

Theorem 3.9. Every gω−closed set is $G_{\beta\omega}$−closed set.
Proof. It is clear, since $Cl_{\beta\omega}(A) \subseteq Cl_{\omega}(A)$. □
The converse of above theorem no need be true.

Example 3.10. In topological space (R,τ), R is the set of real numbers and $\tau = \{\emptyset, R, IR \cup \{2\}\}$, where IR is a set of irrational numbers, the set of rational numbers Q is βω−open set. That is, IR is βω−closed set and thus $Cl_{\beta\omega}(IR) = IR$.

Hence IR is a $G_{\beta\omega}-$closed set. Since Q is not a $\omega-$open set, then IR is not a $\omega-$closed set, that is, $Cl_\omega(IR) \neq IR$. Note that $IR \subseteq IR \cup \{2\}$ and $IR \cup \{2\}$ but $Cl_\omega(IR) \nsubseteq IR \cup \{2\}$, note that for example, $3 \in Cl_\omega(IR)$ and $3 \notin IR \cup \{2\}$. That is, the set IR is not $g\omega-$closed set.

Definition 3.11. A topological space (X,τ) is called anti-locally countable space if each nonempty open set in X is uncountable set.

Lemma 3.12. [7] Let (X,τ) be anti-locally countable space. Then
1. $Int(A) = Int_\omega(A)$ for every $\omega-$closed set A in X.
2. $Cl(A) = Cl_\omega(A)$ for every $\omega-$open set A in X.

Lemma 3.13. For a topological space (X,τ) and $A \subseteq X$, the following hold:
1. $Int_{\beta\omega}(X - A) = X - Cl_{\beta\omega}(A)$.
2. $Cl_{\beta\omega}(X - A) = X - Int_{\beta\omega}(A)$.

Proof. 1. Since $A \subseteq Cl_{\beta\omega}(A)$, then $X - Cl_{\beta\omega}(A) \subseteq X - A$. Since $Cl_{\beta\omega}(A)$ is a $\beta\omega-$closed set then $X - Cl_{\beta\omega}(A)$ is a $\beta\omega-$open set. Then

$$X - Cl_{\beta\omega}(A) = Int_{\beta\omega}[X - Cl_{\beta\omega}(A)] \subseteq Int_{\beta\omega}(X - A).$$

For the other side, let $x \in Int_{\beta\omega}(X - A)$. Then there is $\beta\omega-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\beta\omega-$closed set containing A and $x \notin X - U$. Hence $x \notin Cl_{\beta\omega}(A)$, that is, $x \in X - Cl_{\beta\omega}(A)$.
2. Similar for the part(1). □

Definition 3.14. A subset A of a topological space (X,τ) is called $S_\omega-$open set if $A \subseteq Int_\omega(Cl_\omega(A))$. The complement of $S_\omega-$open set is called $S_\omega-$closed set. The set of all
$S_\omega-$closed sets in X denoted by $S_\omega C(X,\tau)$ and the set of all $S_\omega-$open sets in X denoted by $S_\omega O(X,\tau)$.

Theorem 3.15. Let (X,τ) be anti-locally countable space and $\beta\omega O(X,\tau) = S_\omega O(X,\tau)$. Then
1. $Cl(A) = Cl_\omega(A) = Cl_{\beta\omega}(A)$ for every $\omega-$open set A in X.
2. $Int(A) = Int_\omega(A) = Int_{\beta\omega}(A)$ for every $\omega-$closed set A in X.

Proof. (1) Let A be a $\omega-$open set in X. It is clear from Lemma (3.12) that $Cl(A) = Cl_\omega(A)$ and it is clear that that $Cl_{\beta\omega}(A) \subseteq Cl_\omega(A)$. Now we need to prove that $Cl_\omega(A) \subseteq Cl_{\beta\omega}(A)$. Let $x \notin Cl_{\beta\omega}(A)$. Then there is a $\beta\omega-$open set O in X such that $O \cap A = \emptyset$. Since $\beta\omega O(X,\tau) = S_\omega O(X,\tau)$, then $O \subseteq Int_\omega(Cl_\omega(O)$. Hence $Int_\omega(Cl_\omega(O)$ is a $\omega-$open set
containing x and

$$\begin{aligned} Int_\omega(Cl_\omega(O)) \cap A &= Int_\omega(Cl_\omega(O)) \cap Int_\omega(A) \\ &= Int_\omega[Cl_\omega(O) \cap A] \subseteq Cl_\omega(O) \cap A \\ &\subseteq Cl_\omega(O \cap A) = Cl_\omega(\emptyset) = \emptyset. \end{aligned}$$

That is, $x \notin Cl_\omega(A)$. Hence $Cl_{\beta\omega}(A) \subseteq Cl_\omega(A)$.
(2) Let A be a $\omega-$closed set in X. Then by the part(1), Lemma (3.13) and Theorem (2.2), we get that
$X - Int_{\beta\omega}(A) = Cl_{\beta\omega}(X - A) = Cl_\omega(X - A) = X - Int_\omega(A)$.
That is, $Int_\omega(A) = Int_{\beta\omega}(A)$. By Lemma (3.12), we get that $Int(A) = Int_\omega(A) = Int_{\beta\omega}(A)$. □

Theorem 3.16. Let (X,τ) be anti-locally countable space and $\beta\omega O(X,\tau) = S_\omega O(X,\tau)$. Then X is T_1- space if and only if every $G_{\beta\omega}-$closed set is a $\beta\omega-$closed set in X.

Proof. Necessity: By Theorem (2.7), X is a $T_{1/2}-$ space. Then, by Theorem (3.8), every $G_{\beta\omega}-$closed set is a $\beta\omega-$closed set in X.

Sufficiency: Let $x \in X$ be an arbitrary point in X. By using Theorem (2.7), to prove that X is a T_1- space, we will prove that $\{x\}$ is a closed set in X. Suppose that $\{x\}$ is not closed set in X. Then $A = X - \{x\}$ is not open set. Then X is the only open set containing A and hence $Cl_{\beta\omega}(A) \subseteq X$, that is, A is a $G_{\beta\omega}-$closed set in X. Then, by assumption, A is a $\beta\omega-$closed set. That is, $Cl_{\beta\omega}(A) = A$. Since $X - \{x\}$ is a $\omega-$open set, then by Theorem
(3.15)

$$Cl(A) = Cl_\omega(A) = Cl_{\beta\omega}(A) = A.$$

That is, $\{x\}$ is an open set and this contradicts the fact (X,τ) be anti-locally countable space. Then X is T_1-space. □

We have the following relation for $G_{\beta\omega}-$closed set with the other known sets.

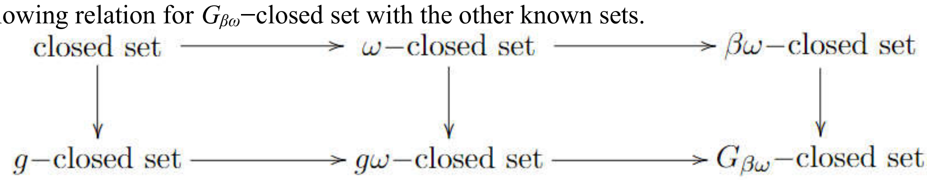

Figure 1:

Theorem 3.17. If A is a $G_{\beta\omega}-$closed set in a topological space (X,τ) and B is a closed set in X then $A \cap B$ is a $G_{\beta\omega}-$closed set.

Proof. Let U be an open subset of X such that $A \cap B \subseteq U$. Since B is a closed set in X then $U \cup (X - B)$ is an open set in X. Since A is a $G_{\beta\omega}-$closed set in X and $A \subseteq U \cup (X - B)$ then $Cl_{\beta\omega}(A) \subseteq U \cup (X - B)$. Hence

$$
\begin{aligned}
Cl_{\beta\omega}(A \cap B) &\subseteq Cl_{\beta\omega}(A) \cap Cl_{\beta\omega}(B) \subseteq Cl_{\beta\omega}(A) \cap Cl(B) \\
&= Cl_{\beta\omega}(A) \cap B \subseteq [U \cup (X - B)] \cap B \\
&\subseteq U \cap B \subseteq U.
\end{aligned}
$$

Thus, $A \cap B$ is a $G_{\beta\omega}-$closed set. □

Theorem 3.18. A subset A of a topological space (X,τ) is a $G_{\beta\omega}-$open if and only if $F \subseteq Int_{\beta\omega}(A)$ whenever $F \subseteq A$ and F is closed subset of (X,τ).

Proof. Let A be a $G_{\beta\omega}-$open subset of X and F be a closed subset of X such that $F \subseteq A$.

Then $X - A$ is a $G_{\beta\omega}-$closed set in X, $X - A \subseteq X - F$ and $X - F$ is an open subset of X. Hence Lemma (3.13), $X - Int_{\beta\omega}(A) = Cl_{\beta\omega}(X - A) \subseteq X - F$, that is, $F \subseteq Int_{\beta\omega}(A)$.

Conversely, suppose that $F \subseteq Int_{\beta\omega}(A)$ where F is a closed subset of X such that $F \subseteq A$. Then for any open subset U of X such that $X - A \subseteq U$, we have $X - U \subseteq A$ and $X-U \subseteq Int_{\beta\omega}(A)$. Then by Lemma(3.13), $X-Int_{\beta\omega}(A) = Cl_{\beta\omega}(X-A) \subseteq U$. Hence $X-A$ is a $G_{\beta\omega}-$closed (i.e., A is a $G_{\beta\omega}-$open set). □

Theorem 3.19. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) then $Cl_{\beta\omega}(A)-A$ contains no nonempty closed set.

Proof. Suppose that $Cl_{\beta\omega}(A) - A$ contains nonempty closed set F. Then

$$F \subseteq Cl_{\beta\omega}(A) - A \subseteq Cl_{\beta\omega}(A).$$

Since $A \subseteq Cl_{\beta\omega}(A)$ then $F \subseteq X - A$ and so $A \subseteq X - F$. Since A is a $G_{\beta\omega}-$closed set and $X - F$ is an open subset of X, then $Cl_{\beta\omega}(A) \subseteq X - F$ and so $F \subseteq X - Cl_{\beta\omega}(A)$. Therefore

$$F \subseteq Cl_{\beta\omega}(A) \cap (X - Cl_{\beta\omega}(A)) = \emptyset$$

and so $F = \emptyset$. Hence $Cl_{\beta\omega}(A) - A$ contains no nonempty closed set. □

Corollary 3.20. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) then $Cl_{\beta\omega}(A)-A$ is a $G_{\beta\omega}-$open set.

Proof. By Theorem (3.19), $Cl_{\beta\omega}(A)-A$ contains no nonempty closed set and it is clear that $\emptyset \subseteq Int_{\beta\omega}(Cl_{\beta\omega}(A) - A)$ then by Theorem (3.18), $Cl_{\beta\omega}(A) - A$ is a $G_{\beta\omega}-$open set. □

Theorem 3.21. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) and $B \subseteq X$. If $A \subseteq B \subseteq Cl_{\beta\omega}(A)$ then B is a $G_{\beta\omega}-$closed set.

Proof. Let U be an open set in X such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since A is a $G_{\beta\omega}-$closed set then $Cl_{\beta\omega}(A) \subseteq U$. Since $B \subseteq Cl_{\beta\omega}(A)$ then

$$Cl_{\beta\omega}(B) \subseteq Cl_{\beta\omega}[Cl_{\beta\omega}(A)] = Cl_{\beta\omega}(A) \subseteq U.$$

That is, B is a $G_{\beta\omega}-$closed set.□

Theorem 3.22. Let A be a $G_{\beta\omega}-$closed subset of a topological space (X,τ). Then $A = Cl_{\beta\omega}(Int_{\beta\omega}(A))$ if and only if $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A$ is a closed set.

Proof. Let $Cl_{\beta\omega}(Int_{\beta\omega}(A))-A$ be a closed set. Since $Int_{\beta\omega}(A) \subseteq A$ and $A \subseteq Cl_{\beta\omega}(A)$, then $Cl_{\beta\omega}(Int_{\beta\omega}(A)) \subseteq Cl_{\beta\omega}(A)$. Then $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq Cl_{\beta\omega}(A) - A$, this implies

$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq X - A \Rightarrow A \subseteq X - (Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A)$.

Since A is a $G_{\beta\omega}-$closed set and $X -(Cl_{\beta\omega}(Int_{\beta\omega}(A))-A)$ is an open set containing A, then $Cl_{\beta\omega}(A) \subseteq X - (Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A)$, this implies

$$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq X - Cl_{\beta\omega}(A).$$

Therefore

$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq Cl_{\beta\omega}(A) \cap (X - Cl_{\beta\omega}(A)) = \emptyset$.

Hence $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A = \emptyset$, that is, $Cl_{\beta\omega}(Int_{\beta\omega}(A)) = A$.

Conversely, if $A = Cl_{\beta\omega}(Int_{\beta\omega}(A))$ then $Cl_{\beta\omega}(Int_{\beta\omega}(A))-A = \emptyset$ and hence $Cl_{\beta\omega}(Int_{\beta\omega}(A))- A$ is a closed set.□

Theorem 3.23. Let Y be an open subset of a topological space (X,τ). If A is a $\beta\omega-$open set in (X,τ) then $A \cap Y$ is a $\beta\omega-$open set in $(Y,\tau|_Y)$.

Proof. Since A be a $\beta\omega-$open set in (X,τ), then $A \subseteq Cl(Int_\omega(Cl(A)))$. Since Y is an open set, then by Theorem (2.1),

$$
\begin{aligned}
A \cap Y &= (A \cap Y) \cap Y \subseteq [(Cl(Int_\omega(Cl(A)))) \cap Y] \cap Y \\
&\subseteq Cl[Int_\omega(Cl(A)) \cap Y] \cap Y = Cl|_Y[Int_\omega(Cl(A)) \cap Y] \\
&= Cl|_Y[Int_\omega(Cl(A)) \cap Int_\omega(Y)] = Cl|_Y[Int_\omega(Cl(A) \cap Y)] \\
&= Cl|_Y[Int_\omega(Cl(A) \cap Y \cap Y)] \subseteq Cl|_Y[Int_\omega(Cl(A \cap Y) \cap Y)] \\
&= Cl|_Y[Int_\omega(Cl|_Y(A \cap Y))] \subseteq Cl|_Y[Int_\omega|_Y(Cl|_Y(A \cap Y))].
\end{aligned}
$$

Therefore $A \cap Y$ is a $\beta\omega-$open set in $(Y,\tau|_Y)$. □

Theorem 3.24. Let Y be an open subset of a topological space (X,τ). If A is a $\beta\omega-$open set in $(Y,\tau|_Y)$ then A is a $\beta\omega-$open set in (X,τ).

Proof. Since A is a $\beta\omega-$open set in $(Y,\tau|_Y)$ and since Y is an open set, then

$$
\begin{aligned}
A &\subseteq Cl|_Y(Int_\omega|_Y(Cl|_Y(A))) = Cl|(Int_\omega|_Y(Cl|_Y(A))) \cap Y \\
&\subseteq Cl|(Int_\omega|_Y(Cl|_Y(A)) \cap Y) = Cl|(Int_\omega(Cl|_Y(A)) \cap Y)
\end{aligned}
$$

$$= Cl|(Int_\omega(Cl|_Y(A) \cap Y)) = Cl|(Int_\omega(Cl|_Y(A)))$$
$$= Cl|(Int_\omega(Cl(A) \cap Y)) \subseteq Cl|(Int_\omega(Cl(A \cap Y)))$$
$$= Cl|(Int_\omega(Cl(A))).$$

Therefore A is a $\beta\omega-$open set in X. □

Theorem 3.25. Let Y be an open subset of a topological space (X,τ) and A be a subset of Y. Then $Cl_{\beta\omega}|_Y(A) = Cl_{\beta\omega}(A) \cap Y$.

Proof. Let $x \in Cl_{\beta\omega}|_Y(A)$ and G be a $\beta\omega-$open set in X containing x. By Theorem (3.23), $G \cap Y$ is a $\beta\omega-$open set in Y containing x and since $x \in Cl_{\beta\omega}|_Y(A)$, then $G \cap A = (G \cap Y) \cap A \; 6= \emptyset$. Then $x \in Cl_{\beta\omega}(A)$ and since $x \in Y$, this implies $x \in Cl_{\beta\omega}(A) \cap Y$. That is, $Cl_{\beta\omega}|_Y(A) \subseteq Cl_{\beta\omega}(A) \cap Y$. On the other side, let $x \in Cl_{\beta\omega}(A) \cap Y$ and O be a $\beta\omega-$open set in Y containing x. By Theorem (3.24), $O = G \cap Y$ for some $\beta\omega-$open set G in X. Since $x \in Cl_{\beta\omega}(A)$, then $G \cap A \; 6= \emptyset$ and so $(G \cap Y) \cap A \; 6= \emptyset$, since $x \in Y$. Hence $O \cap A \; 6= \emptyset$, that is, $x \in Cl_{\beta\omega}|_Y(A)$. Hence $Cl_{\beta\omega}(A) \cap Y \subseteq Cl_{\beta\omega}|_Y(A)$. □

Theorem 3.26. Let Y be an open subspace of a topological space (X,τ) and $A \subseteq Y$. If A is a $G_{\beta\omega}-$closed subset in X then A is a $G_{\beta\omega}-$closed set in Y.

Proof. Let O be an open subset in Y such that $A \subseteq O$. Then $O = U \cap Y$ for some open set U in X and so $A \subseteq U$. Since A is a $G_{\beta\omega}-$closed subset of X, then $Cl_{\beta\omega}(A) \subseteq U$. By Theorem
(3.25),

$$Cl_{\beta\omega}|_Y(A) = Cl_{\beta\omega}(A) \cap Y \subseteq U \cap Y = O.$$

Hence A is a $G_{\beta\omega}-$closed set in Y. □

Theorem 3.27. Let Y be an open subspace of a topological space (X,τ) and $A \subseteq Y$. If A is a $G_{\beta\omega}-$closed subset in Y and Y is $\beta\omega-$closed in X then A is a $G_{\beta\omega}-$closed set in X.

Proof. Let U be an open subset in X such that $A \subseteq U$. Then $A \subseteq U \cap Y$ and $U \cap Y$ is open set in Y. Since A is a $G_{\beta\omega}-$closed subset in Y, then $Cl_{\beta\omega}|_Y(A) \subseteq U \cap Y$. Since Y is an open set in X and it is $\beta\omega-$closed in X then By Theorem (3.25),
$Cl_{\beta\omega}(A) = Cl_{\beta\omega}(A \cap Y) \subseteq Cl_{\beta\omega}(A) \cap Cl_{\beta\omega}(Y) = Cl_{\beta\omega}(A) \cap Y = Cl_{\beta\omega}|_Y(A) \subseteq U \cap Y \subseteq U$.
Hence A is a $G_{\beta\omega}-$closed set in X. □

4 $\beta\omega-$Continuous functions

Definition 4.1. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called $\beta\omega-$*continuous* if $f^{-1}(U)$ is a $\beta\omega-$open set in X for every open set U in Y.

Theorem 4.2. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is $\beta\omega-$continuous if and only if $f^{-1}(F)$ is a $\beta\omega-$closed set in X for every closed set F in Y.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $\beta\omega-$continuous and F be any closed set in Y. Then $f^{-1}(Y - F) = X - f^{-1}(F)$ is a $\beta\omega-$open set in X, that is, $f^{-1}(F)$ is $\beta\omega-$closed set in X. Conversely, suppose that $f^{-1}(F)$ is a $\beta\omega-$closed set in X for every closed set F in Y. Let U be any open set in Y. Then by the hypothesis, $f^{-1}(Y - U) = X - f^{-1}(U)$ is a $\beta\omega-$closed set in X, that is, $f^{-1}(U)$ is a $\beta\omega-$open set in X. Hence f is a $\beta\omega-$continuous. □

Theorem 4.3. Every $\omega-$continuous function is $\beta\omega-$continuous function.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $\omega-$continuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $\omega-$open set in X and hence $f^{-1}(U)$ is a $\beta\omega-$open set in X. That is, f is a $\beta\omega-$continuous function. □

The converse of the last theorem need not be true.

Example 4.4. Let $f : (R,\tau) \to (R,\rho)$ be a function defined by $f(r) = r$, where

$$\tau = \{\emptyset,R\} \text{ and } \rho = \{\emptyset,R,\{2\}\}.$$

The function f is a $\beta\omega-$continuous, since $f^{-1}(\{2\}) = \{2\}$ and $f^{-1}(R) = R$ are $\beta\omega-$open sets in (R,τ). The function f is not $\omega-$continuous, since $f^{-1}(\{2\}) = \{2\}$ is not $\omega-$open set in (R,τ).

Theorem 4.5. If $f : (X,\tau) \to (Y,\rho)$ is a $\beta\omega-$continuous function then for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a $\beta\omega-$open set V in X such that $x \in V$ and $f(V) \subseteq U$.

Proof. Let $x \in X$ and U be any open set in Y containing $f(x)$. Put $V = f^{-1}(U)$. Since f is a $\beta\omega-$continuous then V is a $\beta\omega-$open set in X such that $x \in V$ and $f(V) \subseteq U$.
conversely, Let U be any open set in Y. Let $x \in f^{-1}(U)$. Then $f(x) \in U$ and hence by the hypothesis, there exists a $\beta\omega-$open set V in X such that $x \in V$ and $f(V) \subseteq U$. Hence $x \in V \subseteq f^{-1}(U)$, that is, $f^{-1}(U)$ is a $\beta\omega-$open set in X. That is, f is a $\beta\omega-$continuous. □

Theorem 4.6. Let $f : (X,\tau) \to (Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is a $\beta\omega-$continuous if and only if $f[Cl_{\beta\omega}(A)] \subseteq Cl(f(A))$ for all $A \subseteq X$.

Proof. Let f be a $\beta\omega-$continuous and A be any subset of X. Then $Cl(f(A))$ is a closed set in Y. Since f is a $\beta\omega-$continuous then by Theorem (4.2), $f^{-1}[Cl(f(A))]$ is a $\beta\omega-$closed set in X. That is,

$$Cl_{\beta\omega}\left[f^{-1}[Cl(f(A))]\right] = f^{-1}[Cl(f(A))]$$

Since $f(A) \subseteq Cl(f(A))$ then $A \subseteq f^{-1}[Cl(f(A))]$. This implies,

$$Cl_{\beta\omega}(A) \subseteq Cl_{\beta\omega}\left[f^{-1}[Cl(f(A))]\right] = f^{-1}[Cl(f(A))].$$

Hence $f[Cl_{\beta\omega}(A)] \subseteq Cl(f(A))$.

Conversely, let H be any closed set in Y, that is, $Cl(H) = H$. Since $f^{-1}(H) \subseteq X$. Then by the hypothesis,

$$f\left[Cl_{\beta\omega}[f^{-1}(H)]\right] \subseteq Cl[f(f^{-1}(H))] \subseteq Cl(H) = H.$$

This implies, $Cl_{\beta\omega}[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence $Cl_{\beta\omega}[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\beta\omega-$closed set in X. Therefore f is a $\beta\omega-$continuous. □

Theorem 4.7. Let $f : (X,\tau) \to (Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is $\beta\omega-$continuous if and only if $Cl_{\beta\omega}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for all $B \subseteq Y$.

Proof. Let f be a $\beta\omega-$continuous and B be any subset of Y. Then $Cl(B)$ is a closed set in Y. Since f is a $\omega-$continuous then by Theorem(4.2), $f^{-1}[Cl(B)]$ is a $\beta\omega-$closed set in X. That is,

$$Cl_{\beta\omega}\left[f^{-1}[Cl(B)]\right] = f^{-1}[Cl(B)].$$

Since $B \subseteq Cl(B)$ then $f^{-1}(B) \subseteq f^{-1}[Cl(B)]$. This implies,

$$Cl_{\beta\omega}(f^{-1}(B)) \subseteq Cl_{\beta\omega}\left[f^{-1}[Cl(B)]\right] = f^{-1}[Cl(B)].$$

Hence $Cl_{\beta\omega}(f^{-1}(B)) \subseteq f^{-1}[Cl(B)]$.

Conversely, Let H be any closed set in Y, that is, $Cl(H) = H$. Since $H \subseteq Y$. Then by the hypothesis,

$$Cl_{\beta\omega}(f^{-1}(H)) \subseteq f^{-1}(Cl(H)) = f^{-1}(H).$$

This implies, $Cl_{\beta\omega}[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence $Cl_{\beta\omega}[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\beta\omega-$closed set in X. Hence f is a $\beta\omega-$continuous. □

Theorem 4.8. Let $f : (X,\tau) \to (Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is $\beta\omega-$continuous if and if $f^{-1}(Int(B)) \subseteq Int_{\beta\omega}[f^{-1}(B)]$ for all $B \subseteq Y$.

Proof. Let f be a $\beta\omega-$continuous and B be any subset of Y. Then $Int(B)$ is an open set in Y. Since f is a $\omega-$continuous then $f^{-1}[Int(B)]$ is a $\beta\omega-$open set in X. That is,

$$Int_{\beta\omega}\left[f^{-1}[Int(B)]\right] = f^{-1}[Int(B)].$$

Since $Int(B) \subseteq B$ then $f^{-1}[Int(B)] \subseteq f^{-1}(B)$. This implies,

$$f^{-1}[Int(B)] = Int_{\beta\omega}\left[f^{-1}[Int(B)]\right] \subseteq Int_{\beta\omega}(f^{-1}(B)).$$

Hence $f^{-1}(Int(B)) \subseteq Int_{\beta\omega}[f^{-1}(B)]$.

Conversely, let U be any open set in Y, that is, $Int(U) = U$. Since $U \subseteq Y$. Then by the hypothesis, $f^{-1}(U) = f^{-1}(Int(U)) \subseteq Int_{\beta\omega}[f^{-1}(U)]$.

This implies, $f^{-1}(U) \subseteq Int_{\beta\omega}[f^{-1}(U)]$. Hence $f^{-1}(U) = Int_{\beta\omega}[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a $\beta\omega-$open set in X. Hence f is $\beta\omega-$continuous. □

Definition 4.9. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called *generalized $\beta\omega-$continuous* (simply $G_{\beta\omega}-$continuous) *function*, if $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X for every open set U in Y.

Theorem 4.10. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is $G_{\beta\omega}-$continuous if and only if $f^{-1}(F)$ is a $G_{\beta\omega}-$closed set in X for every closed set F in Y.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $G_{\beta\omega}-$continuous and F be any closed set in Y. Then $f^{-1}(Y - F) = X - f^{-1}(F)$ is a $G_{\beta\omega}-$open set in X, that is, $f^{-1}(F)$ is $G_{\beta\omega}-$closed set in X. Conversely, suppose that $f^{-1}(F)$ is a $G_{\beta\omega}-$closed set in X for every closed set F in Y. Let U be any open set in Y. Then by the hypothesis, $f^{-1}(Y - U) = X - f^{-1}(U)$ is is a $G_{\beta\omega}-$closed set in X, that is, $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. Hence f is a $G_{\beta\omega}-$continuous. □

Theorem 4.11. Every $\beta\omega-$continuous function is $G_{\beta\omega}-$continuous function.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $\beta\omega-$continuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $\beta\omega-$open set in X and by Theorem (3.5), $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. That is, f is a $G_{\beta\omega}-$continuous function. □

The converse of the last theorem need not be true.

Example 4.12. Let $f : (R,\tau) \to (R,\rho)$ be a function defined by $f(r) = r$, where

$$\tau = \{\emptyset, R, R - \{2,3\}\} \text{ and } \rho = \{\emptyset, R, \{2\}\}.$$

The function f is a $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\}) = \{2\}$ and $f^{-1}(R) = R$ are $G_{\beta\omega}-$open sets in (R,τ). The function f is not $\beta\omega-$continuous, since $f^{-1}(\{2\}) = \{2\}$ is not $\beta\omega-$open set in (R,τ).

Theorem 4.13. Let $f : (X,\tau) \to (Y,\rho)$ be a function of a $T_{1/2}-$space (X,τ) into a space (Y,ρ). If f is a $G_{\beta\omega}-$continuous then it is a $\beta\omega-$continuous.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $G_{\beta\omega}-$continuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. Since X is a $T_{1/2}-$space then by Theorem (3.8), $f^{-1}(U)$ is a $\beta\omega-$open set in X. That is, f is a $\beta\omega-$continuous function. □

Theorem 4.14. Every $g\omega-$continuous function is $G_{\beta\omega}-$continuous function.

Proof. Let $f : (X,\tau) \to (Y,\rho)$ be a $g\omega-$continuous function and U be any open set in Y.

Then $f^{-1}(U)$ is a $g\omega-$open set in X and by Theorem (3.9), $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. That is, f is a $G_{\beta\omega}-$continuous function.□
The converse of the last theorem need not be true.

Example 4.15. Let $f : (R,\tau) \rightarrow (R,\rho)$ be a function defined by

$$f(x) = \begin{cases} 2, & x \in IR \\ x, & x \notin IR \end{cases}$$

where

$$\tau = \{\emptyset, R, IR \cup \{2\}\} \text{ and } \rho = \{\emptyset, R, \{2\}\},$$

IR is a set of irrational numbers. The function f is a $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\}) = IR$ and $f^{-1}(R) = R$ are $G_{\beta\omega}-$open sets in (R,τ). The function f is not $g\omega-$continuous, since $f^{-1}(\{2\}) = IR$ is not $g\omega-$open set in (R,τ).

We have the following relation for $G_{\beta\omega}-$continuous function with the other known functions.

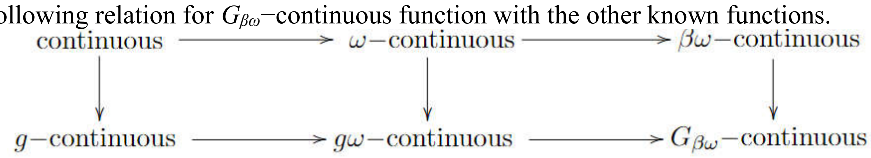

Figure 2:

Theorem 4.16. If $f : (X,\tau) \rightarrow (Y,\rho)$ is a $G_{\beta\omega}-$continuous function then for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a $G_{\beta\omega}-$open set V in X such that $x \in V$ and $f(V) \subseteq U$.

Proof. Let $x \in X$ and U be any open set in Y containing $f(x)$. Put $V = f^{-1}(U)$. Since f is a $G_{\beta\omega}-$continuous then V is a $G_{\beta\omega}-$open set in X such that $x \in V$ and $f(V) \subseteq U$. □

The converse of the last theorem need not be true.

Example 4.17. Let $f : (R,\tau) \rightarrow (R,\rho)$ be a function defined by

$$f(x) = \begin{cases} 2, & x \in \{2,3\} \\ x, & x \notin \{2,3\} \end{cases}$$

where

$$\tau = \{\emptyset, R, R - \{2,3\}\} \text{ and } \rho = \{\emptyset, R, \{2\}\}.$$

The function f is not $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\}) = \{2,3\}$ is not $G_{\beta\omega}-$open set in (R,τ). On the other hand, for all $x \in R$, $\{x\}$ is a $G_{\beta\omega}-$open set in (R,τ).

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