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SEPARATION AXIOMS AND CONNECTEDNESS FOR $\beta\omega-$OPEN SETS

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Abstract:-
Our propose in this paper is to introduce the new classes for separation axioms in topo logical spaces by using $\beta\omega-$open sets and $G_{\beta\omega}-$open sets, called $\beta\omega-$separation axioms and $G_{\beta\omega}-$separation axioms. Furthermore, we introduce the stronger form of connected spaces.

Keywords:-*Open set; Generalized closed set; Connectedness.*

AMS classification: Primary 54A05, 54A10, 54C10

1 INTRODUCTION

In 1970 Levine, [7], introduced the notion of a generalized closed set. A subset A of a space X is called a generalized closed set (simply $g-$closed set) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of a generalized closed set (simply $g-$open set) is called a generalized open set. In 1982 Hdeib [5] introduced the notion of a $\omega-$open sets. A subset A of a space X is called $\omega-$open set if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set. In 1983 the authors [1] introduced the weak form for an open set which is called a $\beta-$open set. A subset A of a space X is called a $\beta-$open set if $A \subseteq Cl(Int(Cl(A)))$. The complement of a β−open set is called a $\beta-$closed set. In 2005 Al-Zoubi [2] introduced the generalization prop erty of $\omega-$open sets. A subset A of a space X is called generalized $\omega-$closed set if $Cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of generalized $\omega-$closed set is called generalized $\omega-$open set, where $Cl_\omega(A)$ is the $\omega-$closure set of A. In 2009 Noiri and Noorani [8] introduced the notion of $\beta\omega-$open set as weak form for a $\omega-$open sets and a $\beta-$open sets. A subset A of a space X is called a $\beta\omega-$open set if $A \subseteq Cl(Int_\omega(Cl(A)))$. The complement of a $\beta\omega-$open set is called a $\beta\omega-$closed set, where $Int_\omega(A)$ is the $\omega-$interior set of A. In 2019 [9] we introduced the notion of $G_{\beta\omega}-$closed set as weak form for a $\beta\omega-$closed sets and a $\beta-$open sets. A subset A of a topological space (X,τ) is called generalized $\beta\omega-$closed (simply $G_{\beta\omega}-$closed) set if $Cl_{\beta\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X,τ). The complement of $G_{\beta\omega}-$closed set is called generalized $\beta\omega-$open (simply $G_{\beta\omega}-$open) set, where $Cl_{\beta\omega}(A)$ is the $\beta\omega-$closure set of A which defined as the intersection of all $\beta\omega-$closed subsets of X containing A. Similar, the $\beta\omega-$interior set of A is defined as the union of all $\beta\omega-$open subsets of X contained in A and is denoted by $Int_{\beta\omega}(A)$.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we introduce the new classes for separation axioms in topological spaces, called $\beta\omega-$separation axioms. Furthermore, the relationship with the other known axioms will be studied. In

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Section 4 we introduce also the new classes for separation axioms in topological spaces, called $G_{\beta\omega}-$separation axioms. Furthermore, the relationship with the other known axioms will be also studied. In Section 5 we introduce the stronger form of connected spaces.

2 Preliminaries

For a topological space (X,τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

A subset of topological space is called a *clopen* set if it is both open and closed set. A topological space (X,τ) is called *0-dimensional space* , [6] if it has a base consisting clopen sets.

Definition 2.1. [6] A topological space (X,τ) is called a *disconnected space* if it is the union of two nonempty subsets A and B such that $Cl(A) \cap B = \emptyset$ and $A \cap Cl(B) = \emptyset$.

Theorem 2.2. [6] A topological space (X,τ) is a disconnected space if and only if it is the union of two disjoint nonempty open subsets.

Theorem 2.3. [6] For a topological space (X,τ) and $A,B \subseteq X$, if B is an open set in X then $Cl(A)capB \subseteq Cl(A \cap B)$.

Theorem 2.4. [7] Every closed set is a $g-$closed set.

Definition 2.5. [7] A topological space (X,τ) is called a *$T_{1/2}-$space* if every $g-$closed set is closed set.

Theorem 2.6. [4] A topological space (X,τ) is *$T_{1/2}-$space* if and only if every singleton set is open or closed set.

Definition 2.7. [6] A topological space (X,τ) is called:

1. *T_0-space* if for two points x 6= $y \in X$ in X, there is open set G in X such that $x \in G$ and $y \in/ G$.
2. *T_1-space* if for two points x 6= $y \in X$ in X, there are two open sets G and U in X such that $x \in G$, $y \in/ G$, $y \in U$ and $x \in/ U$.
3. *T_2-space or Hausdorff space*if for two points x 6= $y \in X$ in X, there are two open sets$G$ and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
4. *regular space* if for each closet set F in X and each $x \in/ F$, there are two open sets G and U in X such that $F \subseteq G$, $x \in/ U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called *T_3-space* if it is regular space and T_1-space.
5. *Normal space* if for each two disjoint closet sets F and M in X, there are two open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called *T_4-space* if it is normal space and T_1-space.

Theorem 2.8. [6] A topological space (X,τ) is T_1-space if and only if every singleton set is closed set.

Theorem 2.9. [6] A topological space (X,τ) is regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is an open set M in X containing x such that $Cl(M) \subseteq N$.

Theorem 2.10. [5] Every open set is $\omega-$open set.

Theorem 2.11. [5] For a topological space (X,τ), the collection of all $\omega-$open sets with a set X forms a topological space.

Theorem 2.12. [8] The union of arbitrary of $\beta\omega-$open sets is $\beta\omega-$open set.

Theorem 2.13. [8] Every $\omega-$open set is $\beta\omega-$open set.

Definition 2.14. [6] A function $f : (X,\tau) \to (Y,\rho)$ of a space (X,τ) into a space (Y,ρ) is called *continuous function* if $f^{-1}(U)$ is an open set in X for every open set U in Y .

Definition 2.15. A function $f : (X,\tau) \to (Y,\rho)$ of a space (X,τ) into a space (Y,ρ) is called:
1. *open function* [6] if $f(U)$ is open set in Y for every open set U in X.
2. *closed function* [6] if $f(U)$ is closed set in Y for every closed set U in X.
3. *$\beta\omega-$continuous function* [9] if $f^{-1}(U)$ is a $\beta\omega-$open set in X for every open set U in Y .

Theorem 2.16. [9] Every $\beta\omega-$open set is $G_{\beta\omega}-$open set.

Theorem 2.17. [9] Let (X,τ) be a topological space. If (X,τ) is a $T_{1/2}-$space then every $G_{\beta\omega}-$closed set in X is $\beta\omega-$closed set in X.

3 *βω*−Separation axioms

Definition 3.1. A topological space (X,τ) is called:
1. *β_ω^2-space* if for two points $x \neq y \in X$ in X, there are two $\beta\omega-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
2. *$\beta\omega-$regular space* if for each closet set F in X and each $x \notin F$, there are two $\beta\omega-$open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called *β_ω^3-space* if it is $\beta\omega-$regular space and T_1-space.
3. *$\beta\omega-$normal space* if for each two disjoint closet sets F and M in X, there are two $\beta\omega-$open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called *β_ω^4-space* if it is $\beta\omega-$normal space and T_1-space.

The proof of the following theorem, Theorem (3.3) and Theorem (3.4) follow from the fact that open sets are $\beta\omega-$open sets.

Theorem 3.2. Every T_2-space is a β_ω^2-space.

Theorem 3.3. Every regular space is a $\beta\omega-$regular space.

Theorem 3.4. Every normal space is a $\beta\omega-$normal space.
The converse of the Theorems (3.2), (3.3) and (3.4) need not be true.

Example 3.5. Let $X = \{1,2,3\}$. The indiscrete topological space (X,T_I), where $T_I = \{\emptyset,X\}$, is β_ω^2-space, $\beta\omega-$regular space and $\beta\omega-$normal space, since all subsets of countable topological space are $\beta\omega-$open sets, but it is not T_2-space, regular space or normal space.

Theorem 3.6. Every β_ω^3-space is a β_ω^2-space.
Proof. Let (X,τ) be a β_ω^3-space and $x \neq y \in X$ be any points in X. Since X is a T_1-space then by Theorem (2.8), $\{x\}$ is a closed set in X and $y \notin \{x\}$. Since X is a $\beta\omega-$regular space then there are two $\beta\omega-$open sets G and U in X such that $x \in \{x\} \subseteq G$, $y \in U$ and $U \cap G = \emptyset$. Hence X is a β_ω^2-space. □

Theorem 3.7. Every β_ω^4-space is a β_ω^3-space.
Proof. Let (X,τ) be a β_ω^4-space. Let F be any closed set in X and $x \notin F$ be any points in X. Since X is a T_1-space then by Theorem (2.8), $\{x\}$ is a closed set in X and $F \cap \{x\} = \emptyset$. Since X is a $\beta\omega-$normal space then there are two $\beta\omega-$open sets G and U in X such that $x \in \{x\} \subseteq G$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence X is a β_ω^3-space. □
We have the following relation.

$$\begin{tikzcd}
T_4-\text{space} \arrow[r] \arrow[d] & T_3-\text{space} \arrow[r] \arrow[d] & T_2-\text{space} \arrow[d] \\
\beta_\omega^4-\text{space} \arrow[r] & \beta_\omega^3-\text{space} \arrow[r] & \beta_\omega^2-\text{space}
\end{tikzcd}$$

Figure 1:

Theorem 3.8. A topological space (X,τ) is a $\beta_\omega{}^2-$space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $\beta\omega-$open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.

Proof. Suppose that (X,τ) is β_ω^2-space. Let $x \in X$ be any point in X and $y \neq x$ be any point in X. Then there are two $\beta\omega-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Take $M = G$ is a $\beta\omega-$open set in X containing x and so $y \notin M \subseteq Cl_{\beta\omega}(M)$.

Conversely, Let $x \neq y \in X$ be any points in X. and By the hypothesis, there is a $\beta\omega-$open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$. Then $X - Cl_{\beta\omega}(M)$ is a $\beta\omega-$open set M in X containing y and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$. Then (X,τ) is β_ω^2-space.

Theorem 3.9. A topological space (X,τ) is a $\beta\omega-$regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is a $\beta\omega-$open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.

Proof. Suppose that (X,τ) is $\beta\omega-$regular space. Let $x \in X$ be any point in X and N be any open set in X containing x. Then $X - N$ is a closed set in X and $x \notin X - N$. Since (X,τ) is $\beta\omega-$regular space then there are two $\beta\omega-$open sets G and U in X such that $X - N \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Take $M = U$ is a $\beta\omega-$open set in X containing x. Then $M = U \subseteq X - G$, this implies,

$$Cl_{\beta\omega}(M) \subseteq Cl_{\beta\omega}(X - G) = X - G \subseteq N.$$

Conversely, Let F be any closed set in X and $x \notin F$. Then $x \in X - F$ and $X - F$ is an open set in X containing x. By the hypothesis, there is a $\beta\omega-$open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(M)$ and $X - Cl_{\beta\omega}(M)$ is a $\beta\omega-$open set in X. Since M is a $\beta\omega-$open set in X containing x and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$, then (X,τ) is $\beta\omega-$regular space. $\square$

Theorem 3.10. A topological space (X,τ) is $\beta\omega-$normal space if and only if for each closed set F in X and for each open set G in X containing F, there is a $\beta\omega-$open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.

Proof. Suppose that (X,τ) is $\beta\omega-$normal space. Let F be any closed set in X and G be any open set in X containing F. Then $X - G$ is a closed set in X and $F \cap (X - G) = \emptyset$. Since (X,τ) is $\beta\omega-$normal space then there are two $\beta\omega-$open sets H and U in X such that $X - G \subseteq U$, $F \subseteq H$ and $U \cap H = \emptyset$. Take $V = H$ is a $\beta\omega-$open set in X containing F. Then $V = H \subseteq X - U$, this implies,

$$Cl_{\beta\omega}(V) \subseteq Cl_{\beta\omega}(X - U) = X - U \subseteq G.$$

Conversely, Let F and M be any two closed sets in X such that $F \cap M = \emptyset$. Then $M \subseteq X - F$ and $X - F$ is an open set in X containing closed set M. By the hypothesis, there is a $\beta\omega-$open set V in X containing M such that $Cl_{\beta\omega}(V) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(V)$ and $X - Cl_{\beta\omega}(V)$ is a $\beta\omega-$open set in X. Since V is a $\beta\omega-$open set in X containing x and $V \cap [X - Cl_{\beta\omega}(V)] = \emptyset$, then (X,τ) is $\beta\omega-$normal space. $\square$

Theorem 3.11. If a function $f : (X,\tau) \to (Y,\rho)$ is $\beta\omega-$continuous injection and Y is a T_2-space then X is a $\beta_\omega{}^2-$space.

Proof. Let Y be a T_2-space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $\beta\omega-$continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega-$open sets in X. Hence X is a $\beta_\omega{}^2-$space. $\square$

A subset of topological space is called a $\beta\omega-$*clopen* set if it is both $\beta\omega-$open and $\beta\omega-$closed set. sets.

Definition 3.12. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called *slightly* $\beta\omega-$*continuous function* if $f^{-1}(U)$ is a $\beta\omega-$clopen set in X for every clopen set U in Y.

Theorem 3.13. Let $f : (X,\tau) \to (Y,\rho)$ be a slightly $\beta\omega-$continuous injection function and Y be 0-dimensional. If Y is a T_2-space then X is a $\beta_\omega{}^2-$space.

Proof. Let Y be a T_2-space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Since Y is 0-dimensional space there are two clopen sets G_1 and U_1 in Y such that

$$f(x) \subseteq G_1 \subseteq G \qquad \text{and} \qquad f(y) \subseteq U_1 \subseteq U.$$

Then $x \subseteq f^{-1}(G_1) \subseteq f^{-1}(G)$ and $y \subseteq f^{-1}(U_1) \subseteq f^{-1}(U)$.
and $f^{-1}(G_1) \cap f^{-1}(U_1) \subseteq f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset$.
Since G_1 and U_1 are clopen sets in Y and f is a slightly $\beta\omega-$continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega-$open sets in X. Hence X is a β_ω^2-space. $\square$

Theorem 3.14. Let $f : (X,\tau) \to (Y,\rho)$ be $\beta\omega-$continuous injection function. If f is an open (or closed) function and Y is a regular space then X is a $\beta\omega-$regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x. Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y. Since Y is a regular space then by Theorem(2.9), there is an open set M in Y containing $f(x)$ such that $Cl(M) \subseteq f(U)$. Since f is a $\beta\omega-$continuous then $V = f^{-1}(M)$ is a $\beta\omega-$open set in X containing x. Since f is injection then

$$f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Then

$$Cl_{\beta\omega}(V) = Cl_{\beta\omega}[f^{-1}(M)] \subseteq f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Hence by Theorem (3.9), X is a $\beta\omega-$regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y. Since Y is a regular space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(x) \in U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $x \notin f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $\beta\omega-$open in X. Hence X is a $\beta\omega-$regular space.

Theorem 3.15. Let $f : (X,\tau) \to (Y,\rho)$ be slightly $\beta\omega-$continuous injection and Y is 0-dimensional space. If f is an open (or closed) function then X is a $\beta\omega-$regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x. Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y. Since Y is a 0-dimensional space then there is a clopen set V in Y such that $f(x) \in V \subseteq f(U)$. Since f is injection then $x \in f^{-1}(V) \subseteq U$. Since f is a $\beta\omega-$continuous then $f^{-1}(V)$ is a $\beta\omega-$clopen set in X containing x. Hence

$$Cl_{\beta\omega}(f^{-1}(V)) = f^{-1}(V) \subseteq U.$$

Hence by Theorem (3.9), X is a $\beta\omega-$regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y. Then $f(x) \in Y - f(F)$ and $Y - f(F)$ is an open set in Y. Since Y is a 0-dimensional space then there is a clopen set V in Y such that $f(x) \in V \subseteq Y - f(F)$. Since f is injection then

$$x \in f^{-1}(V) \subseteq f^{-1}[Y - f(F)] \subseteq X - F.$$

Since f is a slightly $\beta\omega-$continuous then $f^{-1}(V)$ is a $\beta\omega-$clopen set in X containing x and $X - f^{-1}(V)$ is a $\beta\omega-$clopen set in X such that $F \subseteq X - f^{-1}(V)$ Hence X is a $\beta\omega-$regular space. □

Theorem 3.16. Let $f : (X,\tau) \to (Y,\rho)$ be $\beta\omega-$continuous injection function. If f is closed function and Y is a normal space then X is a $\beta\omega-$normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. since Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $\beta\omega-$open in X. Hence X is a $\beta\omega-$normal space. □

Theorem 3.17. Let $f : (X,\tau) \to (Y,\rho)$ be slightly $\beta\omega-$continuous injection and Y is 0 dimensional space. If f is a closed function and Y is a normal space then X is a $\beta\omega-$normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since Y is a 0-dimensional space then for every $g \in f(F)$ and $u \in f(H)$ there are clopen sets U_u and G_g in Y such that

$$u \in U_u \subseteq U \quad \text{and} \quad g \in G_g \subseteq G.$$

Then $f(H) \subseteq \cup\{U_u : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\} \subseteq U$
and $f(F) \subseteq \cup\{G_g : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\} \subseteq G$.
This implies,
$H \subseteq \cup\{f^{-1}(U_u) : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\} \subseteq f^{-1}(U)$
and
$F \subseteq \cup\{f^{-1}(G_g) : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\} \subseteq f^{-1}(G)$.
Since f is a slightly $\beta\omega-$continuous then $f^{-1}(U_u)$ and $f^{-1}(G_g)$ are $\beta\omega-$open in X for all $g \in f(F)$ and $u \in f(H)$. So that

$$M = \cup\{f^{-1}(U_u) : u \in f(H)\} \text{ and } \qquad N = \cup\{f^{-1}(G_g) : g \in f(F)\}$$

are $\beta\omega-$open in X and
$M \cap N \subseteq f^{-1}(U) \cap f^{-1}(G) \subseteq f^{-1}(U \cap G) = f^{-1}(\emptyset) = \emptyset$.
Hence X is a $\beta\omega-$normal space.

□

4 $G_{\beta\omega}-$Separation axioms

Definition 4.1. A topological space (X,τ) is called:

1. $G^2_{\beta\omega}$*−space* if for two points $x \neq y \in X$ in X, there are two $G_{\beta\omega}-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
2. $G_{\beta\omega}$*−regular space* if for each closet set F in X and each $x \notin F$, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called $G^3_{\beta\omega}$*−space* if it is $G_{\beta\omega}-$regular space and T_1-space.
3. $G_{\beta\omega}$*−normal space* if for each two disjoint closet sets F and M in X, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X,τ) is called $G^4_{\beta\omega}$*−space* if it is $G_{\beta\omega}-$normal space and T_1-space.

It is clear that every $\beta_\omega{}^2-$space is a $G^2{}_{\beta\omega}-$space, every $\beta\omega-$regular space is a $G_{\beta\omega}-$regular space and every $\beta\omega-$normal space is a $G_{\beta\omega}-$normal space.

Theorem 4.2. Every $G^3{}_{\beta\omega}-$space is a $G^2{}_{\beta\omega}-$space.
Proof. Similar for Theorem (3.6).

Theorem 4.3. Every $G^4{}_{\beta\omega}-$space is a $G^3{}_{\beta\omega}-$space.
Proof. Similar for Theorem (3.7). □

Theorem 4.4. Let (X,τ) be a $T_{1/2}-$space. If X is a $G^2{}_{\beta\omega}-$space then X is a $\beta_\omega{}^2-$space.
Proof. For two points $x \neq y \in X$ in X, since X is a $G^2{}_{\beta\omega}-$space, there are two $G_{\beta\omega}-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}-$space, then by Theorem (2.17), G and U are $\beta\omega-$open sets in X. Hence X is a $\beta^2_\omega-$space. □

Theorem 4.5. Let (X,τ) be a $T_{1/2}-$space. If X is a $G_{\beta\omega}-$regular space then X is a $\beta\omega-$regular space
Proof. For each closet set F in X and each $x \notin F$, since X is a $G_{\beta\omega}-$regular space, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}-$space, then by Theorem (2.17), G and U are $\beta\omega-$open sets in X. Hence X is a $\beta\omega-$regular space. □

Corollary 4.6. Every $G^3{}_{\beta\omega}-$space is a $\beta^3_\omega-$space.
Proof. Use above theorem, since every T_1-space is $T_{1/2}-$space. □

Theorem 4.7. Let (X,τ) be a $T_{1/2}-$space. If X is a $G_{\beta\omega}-$normal space then X is a $\beta\omega-$normal space
Proof. For each two disjoint closet sets F and M in X, since X is a $G_{\beta\omega}-$normal space, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}-$space then by Theorem (2.17), G and U are $\beta\omega-$open sets in X. Hence X is a $\beta\omega-$normal space. □

Corollary 4.8. Every $G^4{}_{\beta\omega}-$space is a $\beta_\omega{}^4-$space.
Proof. Use above theorem, since every T_1-space is $T_{1/2}-$space. □
We have the following relation.

$$\begin{tikzcd}
T_4-\text{space} \arrow[r] \arrow[d] & T_3-\text{space} \arrow[r] \arrow[d] & T_2-\text{space} \arrow[d] \\
\beta^4_\omega-\text{space} \arrow[r] \arrow[d, leftrightarrow] & \beta^3_\omega-\text{space} \arrow[r] \arrow[d, leftrightarrow] & \beta^2_\omega-\text{space} \arrow[d] \\
G^4_{\beta\omega}-\text{space} \arrow[r] & G^3_{\beta\omega}-\text{space} \arrow[r] & G^2_{\beta\omega}-\text{space}
\end{tikzcd}$$

Figure 2:

Theorem 4.9. A topological space (X,τ) is $G^2{}_{\beta\omega}-$space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $G_{\beta\omega}-$open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.
Proof. Similar for Theorem (3.8). □

Theorem 4.10. A topological space (X,τ) is $G_{\beta\omega}-$regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is a $G_{\beta\omega}-$open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.
Proof. Similar for Theorem (3.9). □

Theorem 4.11. A topological space (X,τ) is $G_{\beta\omega}-$normal space if and only if for each closed set F in X and for each open set G in X containing F, there is a $G_{\beta\omega}-$open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.
Proof. Similar for Theorem (3.10). □

Theorem 4.12. If a function $f : (X,\tau) \to (Y,\rho)$ is $G_{\beta\omega}-$continuous injection and Y is a T_2-space then X is a $G^2{}_{\beta\omega}-$space.
Proof. Let Y be a T_2-space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $G_{\beta\omega}-$continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $G_{\beta\omega}-$open sets in X. Hence X is a $G^2{}_{\beta\omega}-$space. □

Theorem 4.13. Let $f : (X,\tau) \to (Y,\rho)$ be $G_{\beta\omega}-$continuous injection function. If f is an open (or closed) function and Y is a regular space then X is a $G_{\beta\omega}-$regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x. Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y. Since Y is a regular space then by Theorem(2.9), there is an open set M in Y containing $f(x)$ such that $Cl(M) \subseteq f(U)$. Since f is a $G_{\beta\omega}-$continuous then $V = f^{-1}(M)$ is a $G_{\beta\omega}-$open set in X containing x. Since f is injection then

$$f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Hence

$$Cl_{\beta\omega}(V) = Cl_{\beta\omega}[f^{-1}(M)] \subseteq f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Then by Theorem (4.10), X is a $G_{\beta\omega}-$regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x /\in F$. Then $f(x) \in/ f(F)$ and $f(F)$ is a closed set in Y. Since Y is a regular space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(x) \in U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $x /\in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $G_{\beta\omega}-$continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $G_{\beta\omega}-$open in X. Hence X is a $G_{\beta\omega}-$regular space.

Theorem 4.14. Let $f : (X,\tau) \to (Y,\rho)$ be $G_{\beta\omega}-$continuous injection function. If f is closed function and Y is a normal space then X is a $G_{\beta\omega}-$normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. since Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $G_{\beta\omega}-$continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $G_{\beta\omega}-$open in X. Hence X is a $G_{\beta\omega}-$normal space. □

5 $\beta\omega-$Connectedness property

Definition 5.1. Let (X,τ) be a topological space and A,B be two nonempty subsets of X. The sets A and B are called a *$\beta\omega-$separated sets* if $Cl_{\beta\omega}(A) \cap B = \emptyset$ and $A \cap Cl_{\beta\omega}(B) = \emptyset$.

Remark 5.2. Let (X,τ) be a topological space. Then

1. Any $\beta\omega-$separated sets are disjoint sets, since $A \cap B \subseteq A \cap Cl_{\beta\omega}(B) = \emptyset$.
2. Any two nonempty $\beta\omega-$closed sets in X are $\beta\omega-$separated if they are disjoint sets.

Definition 5.3. A topological space (X,τ) is called a *$\beta\omega-$disconnected space* if it is the union of two $\beta\omega-$separated sets. Otherwise A (X,τ) is called a *$\beta\omega-$connected space*.

Example 5.4. Any a countable topological space (X,τ) is a $\beta\omega-$disconnected space if X has more that one point.

The proof of the following theorem is clear since $Cl_{\beta\omega}(A) \subset Cl(A)$.

Theorem 5.5. Every disconnected space is a $\beta\omega-$disconnected space.

The converse of the above theorem need not be true.

Example 5.6. In the topological space (X,T), where $T = \{\emptyset,X\}$ and $X = \{a,b\}$, is $\beta\omega-$disconnected space but it is a connected space.

Theorem 5.7. A topological space (X,τ) is a $\beta\omega-$disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega-$open sets.

Proof. Suppose that (X,τ) is a $\beta\omega-$disconnected space. Then X is the union of two $\beta\omega-$separated sets, that is, there are two nonempty subsets A and B of X such that

$Cl_{\beta\omega}(A) \cap B = \emptyset$, $A \cap Cl_{\beta\omega}(B) = \emptyset$ and $A \cup B = X$.

Take $G = X - Cl_{\beta\omega}(A)$ and $H = X - Cl_{\beta\omega}(B)$. Then G and H are $\beta\omega-$open sets. Since B 6= $\emptyset$ and $Cl_{\beta\omega}(A) \cap B = \emptyset$, then $B \subseteq X - Cl_{\beta\omega}(A)$, that is, $G = X - Cl_{\beta\omega}(A)$ 6= $\emptyset$. Similar H 6= $\emptyset$. Since $Cl_{\beta\omega}(A) \cap B = \emptyset$, $A \cap Cl_{\beta\omega}(B) = \emptyset$ and $A \cup B = X$, then

$$X - (G \cap H) = (X - G) \cup (X - H) = [Cl_{\beta\omega}(A)] \cup [Cl_{\beta\omega}(B)] = X.$$

That is, $G \cap H = \emptyset$.

Conversely, suppose that (X,τ) is the union of two disjoint nonempty $\beta\omega-$open subsets, say G and H. Take $A = X - G$ and $B = X - H$. Then A and B are $\beta\omega-$closed sets, that is, $Cl_{\beta\omega}(A) = A$ and $Cl_{\beta\omega}(B) = B$. Since H 6= $\emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, A 6= $\emptyset$. Similar B 6= $\emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

That is, (X,τ) is a $\beta\omega-$disconnected space. □

Corollary 5.8. A topological space (X,τ) is a $\beta\omega-$disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega-$closed subsets.

Proof. Suppose that (X,τ) is a $\beta\omega-$disconnected space. Then by Theorem (5.7), (X,τ) is the union of two disjoint nonempty $\beta\omega-$open subsets, say G and H. Then $X-G$ and $X-H$ are $\beta\omega-$closed subsets. Since G 6= $\emptyset$, H 6= $\emptyset$ and $X = G \cup H$ then $X-G$ 6= $\emptyset$, $X-H$ 6= $\emptyset$ and

$$(X-G) \cap (X-H) = X-(G \cup H) = X-X = \emptyset.$$

Since $G \cap H = \emptyset$ then

$$(X-G) \cup (X-H) = X-(G \cap H) = X-\emptyset = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega-$closed subsets.

Conversely, suppose that (X,τ) is the union of two disjoint nonempty $\beta\omega-$closed subsets, say G and H. Take $A = X-G$ and $B = X-H$. Then A and B are $\beta\omega-$open sets. Since H 6= $\emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X-G = A$, that is, A 6= $\emptyset$. Similar B =6 $\emptyset$. Since $G\cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X-G) \cap (X-H) = X-(G \cup H) = X-X = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X-G) \cup (X-H) = X-(G \cap H) = X-\emptyset = X.$$

Then by Theorem (5.7), (X,τ) is a $\beta\omega-$disconnected space. □

Theorem 5.9. A topological space (X,τ) is a $\beta\omega-$connected space if there is no nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed.

Proof. Suppose that (X,τ) is a $\beta\omega-$connected space. Let A be a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. Then $X-A$ is a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. Since $A \cup (X-A) = X$, then by Theorem (5.7), X is a $\beta\omega-$disconnected space and this a contradiction. So there is no nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed set.

Conversely, suppose that (X,τ) is a $\beta\omega-$disconnected space. Then by Theorem (5.7), X is the union of two disjoint nonempty $\beta\omega-$open subsets, say A and B. Then $X-B = A$ is $\beta\omega-$closed subset of X. Since B 6= $\emptyset$ and $X = A \cup B$ then A is a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. This is a contradiction with the hypothesis. Hence (X,τ) is a $\beta\omega-$connected space. □

Theorem 5.10. Let $f : (X,\tau) \to (Y,\rho)$ be a $\beta\omega-$continuous surjection function. If X is a $\beta\omega-$connected space then Y is connected space.

Proof. Suppose that Y is a disconnected space. Then by Theorem (5.7), Y is the union of two disjoint nonempty open subsets, say G and H. Since f is a $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega-$open sets in X. Since G 6= $\emptyset$, H =6 $\emptyset$ and f is a surjection then $f^{-1}(H)$ 6= $\emptyset$ and $f^{-1}(G)$ 6= $\emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$ then

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset$$

and

$$f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(Y) = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega-$open subsets, that is, X is a $\beta\omega-$disconnected space. This is a a contradiction. Hence Y is a connected space. □

Theorem 5.11. Let $f : (X,\tau) \to (Y,\rho)$ be a slightly $\beta\omega-$continuous surjection function. If X is a $\beta\omega-$connected space then Y is connected space.

Proof. Suppose that Y is a disconnected space. Then by Theorem (2.2), Y is the union of two disjoint nonempty open subsets, say G and B. Then G and B are clopen sets in Y. Since f is a slightly $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega-$open sets in X. Since G 6= $\emptyset$, H 6= $\emptyset$ and f is a surjection then $f^{-1}(H)$ =6 $\emptyset$ and $f^{-1}(G)$ =6 $\emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$ then $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Hence X is the union of two disjoint nonempty $\beta\omega-$open subsets, that is, X is a $\beta\omega-$disconnected space. This is a a contradiction. Hence Y is a connected space. □

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