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ON PREOPENNESS PROPERTY IN GRILL TOPOLOGICAL SPACES

Amin *Saif*[1*], *Mohammed Al-Hawmi*[2], *Mated Abulwahab*[3],

*[*1]Department of Mathematics, Faculty of Sciences, Taiz University, Taiz, Yemen*

[23]Department of Mathematics,Faculty of Education, University of Saba Region, Mareb, Yemen

m.alhomi2011@gmail.com, majedabuaceamh@gmail.com

Corresponding Author:-

Email: alsanawyamin@yahoo.com

Abstract:-

The class of a $N-$preopen sets was introduced by [1] in topological spaces. In this paper we extend the notion of this class into grill topological spaces by giving the concept of G_N-preopen sets as a strong form of the class of a $N-$preopen sets.

Keywords:-*Preopen sets; Grill topological spaces.*

AMS classification: Primary 54C08, 54C05

1 INTRODUCTION

For a topological space (X,τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively. In 1982 Hdeib [10], introduced the no tions of $w-$open set. A subset A of a topological space (X,τ) is called $\omega-$open set if for each $x \in A$, there is open set U_x containing x such that $U_x - A$ is a countable set. The comple ment of $\omega-$open set is called $\omega-$closed set. In 1982 Mashhour [2], introduced the notion of a preopen set. A subset A of a topological space X is called a preopen set if $A \subseteq Int(Cl(A))$. The complement of preopen set is called preclosed set. In 2009 Al-Omari and Noiri [1], used the notions of preopen sets, $w-$open sets and finiteness property to introduce the notions of N−preopen sets. A subset A of topological space (X,τ) is called a N−preopen set if for each $x \in A$, there exists a preopen set U_x containing x such that $U_x - A$ is a finite set. The complement of N−preopen set is called N−preclosed set.

The idea of grill on a topological space, given by Choquet [6], goes as follows: A non-null collection G of subsets of a topological spaces X is said to be a *grill* on X if

(i) $A \in$ G and $A \subseteq B \Longrightarrow B \in$ G

(ii) $A,B \subseteq X$ and $A \cup B \in$ G $\Longrightarrow A \in$ G or $B \in$ G.

For a topological space X, the operator $\Phi : P(X) \to P(X)$ from the power set $P(X)$ of X to $P(X)$ was first defined in [11] in terms of grill; the latter concept being defined by Choquet [6] several decades back. Interestingly, it is found from subsequent investigations that the notion of grills as an appliance like nets and filters, turns out to be extremely useful towards the study of certain specific topological problems (see for instance [8], [9] and [12]). For a grill G on a topological space X, an operator from the power set $P(X)$ of X to $P(X)$ was defined in [3] in the following manner : For any $A \in P(X)$, $\Phi(A) = \{x \in X : U \cap A \in$ G, for each open neighborhood U of $x\}$.

Then the operator $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [3] to be a Kuratowski closure operator, defining a unique topology τ_G on X such that $\tau \subseteq \tau_G$. If (X,τ) is a topological space and G is a grill on X then the triple $(X,\tau,$G$)$ will be called a *grill topological space*. Under the notion of grill topological space and its operators above, several authors defined and investigated the notions in this part. In 2010 Hatir and Jafari [4], introduced the notions of G−preopen set in grill topological spaces. A subset A of a grill topological space $(X,\tau,$G$)$ is called a G−preopen set if $A \subseteq Int(\Psi(A))$. The complement of G−preopen set is called G−preclosed set.

This paper is organized as follows. In Section 2 we introduce the concept of G_N−preopen sets. Furthermore, the relationship with the other known sets will be studied. In Section 3 we define and investigate the interior operator and the closure operator via G_N−preopen sets. In Section 4 we study G_N−preopen sets with the relative topologies and the product topology.

A topological space (X,τ) is called: T_1-space [5] if for each disjoint point $x \neq y \in X$, there are two open sets G and H in X such that $x \in H$, $y \in G$, $x \notin G$ and $y \notin H$.

Theorem 1.1. [5] A topological space (X,τ) is T_1-space if and only if every singleton set is closed set.

Theorem 1.2. [2] Let A and B be two subsets in a topological space (X,τ). If A is a preopen set in X and B is an open set in X then $A \cap B$ is a preopen set in X.

Theorem 1.3. [2] Let A and Y be two subsets in a topological space (X,τ). If A is a preopen set in X and Y is an open set in X then $A \cap Y$ is a preopen set in $(Y,\tau|_Y)$.

Theorem 1.4. [2] Let Y be an open subset of a topological space (X,τ). If A is a preopen set in $(Y,\tau|_Y)$ then $A = G \cap Y$ for some a preopen set G in X.

Theorem 1.5. [1] The union of any family of N−preopen sets is N−preopen set.

Theorem 1.6. [3] Let $(X,\tau,$G$)$ be a grill topological space. Then for $A,B \subseteq X$, the following properties hold:

1. $A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$.
2. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.
3. $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.
4. If $U \in \tau$ then $U \cap \Phi(A) \subseteq \Phi(U \cap A)$.

Theorem 1.7. [4] Every G−preopen set in a grill topological space $(X,\tau,$G$)$ is preopen set.

Theorem 1.8. [4] The union of two G−preopen sets in a grill topological space $(X,\tau,$G$)$ is G−preopen set.

2 G_N−Preopen sets

Definition 2.1. A subset A of a grill topological space $(X,\tau,$G$)$ is called a G_N−preopen set if for each $x \in A$, there exists a G−preopen set U_x containing x such that $U_x - A$ is a finite set. The complement of G_N−preopen set is called G_N−preclosed set.

The set of all G_N−preopen sets in X denoted by $G_N\,O(X,\tau,$G$)$ and the set of all G_N−preclosed sets in X denoted by $G_N\,C(X,\tau,$G$)$.

Example 2.2. For a grill topological space $(X,\tau,$G$)$ with a finite set X,

$$\mathrm{G_N}\,O(X,\tau,\mathrm{G}) = \mathrm{G_N}\,C(X,\tau,\mathrm{G}) = P(X),$$

where $P(X)$ is the power of X (i.e., the collection of all subsets of X).

Example 2.3. Let $(\mathrm{R},\tau_u,\mathrm{G})$ be a grill topological space with the real usual topology τ_u on the set of real numbers R and G $= P(\mathrm{R}) - \{\emptyset\}$. The rational set Q is a $\mathrm{G_N}$−preopen set.

Theorem 2.4. Let (X,τ,G) be a grill topological space. Then:
1. X and $\emptyset$ are $\mathrm{G_N}$−preopen sets.
2. Any subset of X with a finite complement is $\mathrm{G_N}$−preopen set.

Proof. (1) Since $X = Int(\Psi(X))$, then X is a G-preopen set and hence for each $x \in X$, X is a G−preopen set containing x and $X - X = \emptyset$ is a finite set. That is, X is a $\mathrm{G_N}$−preopen set. Similar for $\emptyset$.
(2) Let A be a subset of a grill topological space (X,τ,G) with a finite complement. That is, $X - A$ is a finite set. Since X is G-preopen set, then A is a $\mathrm{G_N}$−preopen set. □

Theorem 2.5. Every G-preopen set is a $\mathrm{G_N}$−preopen set.
Proof. Let A be G-preopen subset of a grill topological space (X,τ,G). Hence for each $x \in A$, A is a G−preopen set containing x and $A - A = \emptyset$ is a finite set. That is, A is a $\mathrm{G_N}$−preopen set.□
The converse of above theorem need not be true.

Example 2.6. Let (X,τ,G) be a grill topological space, where $X = \{a,b,c\}$, $\tau = \{\emptyset,X,\{a\},\{a,b\}\}$ and $\mathrm{G} = P(X) - \{\emptyset\}$. The set $\{c\}$ is a $\mathrm{G_N}$−preopen set but it is not a G-preopen set.

Theorem 2.7. Every $\mathrm{G_N}$−preopen set is a N−preopen set.
Proof. Let A be a $\mathrm{G_N}$−preopen subset of a grill topological space (X,τ,G). Hence for each $x \in A$, there is a G−preopen set $U_x$containing x such that $U_x - A$ is a finite set. Then for each $x \in A$, U_x is a preopen set containing x, that is, A is a N−preopen set. □
The converse of above theorem need not be true.

Example 2.8. Let $(\mathrm{R},\tau,\mathrm{G})$ be a grill topological space, where $\tau = \{\emptyset,\mathrm{R}\}$ and $\mathrm{G} = \{\mathrm{R}\}$. For $a \in \mathrm{R}$, the set $\{a\}$ is a preopen set, since
$$\{a\} \subseteq Int(Cl(\{a\})) = Int(\mathrm{R}) = \mathrm{R}.$$
Then $\{a\}$ is a N−preopen set. If U_a is a G−preopen set containing a and $U_a - \{a\}$ is a finite then U_a is a finite. Hence $Int(\Psi(U_a)) = Int(U_a) = \emptyset$ and this is contradiction. Then $\{a\}$ is not $\mathrm{G_N}$−preopen set.

Theorem 2.9. The union of any family of $\mathrm{G_N}$−preopen sets is $\mathrm{G_N}$−preopen set.
Proof. Let A_λ be a $\mathrm{G_N}$−preopen subset of a grill topological space (X,τ,G) for all $\lambda \in \Delta$. Let $x \in \bigcup_{\lambda\in\Delta} A_\lambda$ be an arbitrary point. Then there is at least $\lambda_o \in \Delta$ such that $x \in A_{\lambda_o}$. Since A_{λ_o} is a $\mathrm{G_N}$−preopen set then there exists a G−preopen set U_x containing x such that $U_x - A_{\lambda_o}$ is a finite set. Since $A_{\lambda_o} \subseteq \bigcup_{\lambda\in\Delta} A_\lambda$, then
$$U_x - \bigcup_{\lambda\in\Delta} A_\lambda \subseteq U_x - A_{\lambda_o}.$$
Hence $U_x - \bigcup_{\lambda\in\Delta} A_\lambda$ is a finite set. That is, $\bigcup_{\lambda\in\Delta} A_\lambda$ is $\mathrm{G_N}$−preopen set. □
The intersection of two $\mathrm{G_N}$−preopen sets need not be $\mathrm{G_N}$−preopen set.

Example 2.10. Let $(\mathrm{R},\tau_u,\mathrm{G})$ be a grill topological space with the real usual topology τ_u on the set of real numbers R and $\mathrm{G} = \{\mathrm{R}\}$. Let $A = IR$ and $B = Q \cup \{\pi\}$, where Q is the set of rational numbers and IR is the set of irrational numbers. Since
$$\Psi(A) = \Psi(IR) = \mathrm{R} \text{ and } \Psi(B) = \Psi(Q \cup \{\pi\}) = \mathrm{R},$$
then A and B are G−preopen sets and so $\mathrm{G_N}$−preopen sets. The set $A \cap B = \{\pi\}$ is not a $\mathrm{G_N}$−preopen set. If U_π is a G−preopen set containing π and $U_\pi - \{\pi\}$ is a finite then U_π is a finite. Hence $Int(\Psi(U_\pi)) = Int(U_\pi) = \emptyset$ and this is contradiction. Then $\{\pi\}$ is not $\mathrm{G_N}$−preopen set.
A subset A of grill topological space (X,τ,G) is called a *dense* in X if $Cl(A) = X$. A grill topological space (X,τ,G) is called submaximal grill topological space if every dense subset of X is open set.

Lemma 2.11. [7] Let (X,τ) be a submaximal space. Then every preopen set is open set.

Lemma 2.12. Let (X,τ,G) be a submaximal grill topological space. Then every G−preopen set is open set.
Proof. By Lemma(2.11), every preopen set is open set and by Theorem(2.7), every $\mathrm{G_N}$−preopen set is open set. □
A subset A of topological space (X,τ) is called a *dense* in X if $Cl(A) = X$. A topological space (X,τ) is called submaximal space if every dense subset of X is open set.

Theorem 2.13. Let (X,τ,G) be a submaximal grill topological space. Then $[X,\mathrm{G_N}\,O(X,\tau,\mathrm{G})]$ is a topological space.
Proof.
(1) It is clear that $\emptyset$ and X are $\mathrm{G_N}$−preopen sets.

(2) Let A and B be two G_N−preopen sets and $x \in A \cap B$ be arbitrary point. Then there are two G−preopen sets U_x and G_x containing x such that $U_x - A$ and $G_x - B$ are finite sets.
By lemma above, $U_x \cap G_x$ is G−preopen set containing x and

$$\begin{aligned}(U_x \cap G_x) - (A \cap B) &= (U_x \cap G_x) \cap [X - (A \cap B)] \\ &= (U_x \cap G_x) \cap [(X - A) \cup (X - B)] \\ &= [(U_x \cap G_x) \cap (X - A)] \cup [(U_x \cap G_x) \cap (X - B)] \\ &\subseteq [U_x \cap (X - A)] \cup [G_x \cap (X - B)] \\ &= (U_x - A) \cup (G_x - B).\end{aligned}$$

That is, $(U_x \cap G_x) - (A \cap B)$ is a finite and hence $A \cap B$ is a G_N−preopen set.
(3) Let A_λ be a G_N−preopen subset of a grill topological space (X,τ,G) for all $\lambda \in \Delta$. By Theorem(2.9), $\bigcup_{\lambda \in \Delta} A_\lambda$ is a G_N −preopen set. □

Lemma 2.14. The intersection of an open set and a G−preopen set is a G−preopen set.
Proof. Let A be an open set and B be a G−preopen set in a grill topological space (X,τ,G). Then $B \subseteq Int(\Psi(B))$. Since $\tau \subseteq \tau_G$ then

$$\begin{aligned}A \cap B &\subseteq A \cap [Int(\Psi(B))] \\ &= Int(A) \cap Int(\Psi(B)) \\ &= Int(A \cap \Psi(B)) \\ &\subseteq Int(\Psi(A \cap B))\end{aligned}$$

Theorem 2.15. The intersection of an open set and G_N−preopen set is a G_N−preopen set.
Proof. Let A be an open set and B be a G_N−preopen set in a grill topological space (X,τ,G). Let $x \in A \cap B$ be arbitrary point. Then there is a G−preopen set U_x containing x such that $U_x - B$ is a finite sets. By lemma above, $A \cap U_x$ is a G−preopen set and we get that

$$\begin{aligned}(A \cap U_x) - (A \cap B) &= (A \cap U_x) \cap [X - (A \cap B)] \\ &= (A \cap U_x) \cap [(X - A) \cup (X - B)] \\ &= [(A \cap U_x) \cap (X - A)] \cup [(A \cap U_x) \cap (X - B)] \\ &= (A \cap U_x) \cap (X - B) = (A \cap U_x) - B \\ &\subseteq U_x - B.\end{aligned}$$

That is, $(A \cap U_x) - (A \cap B)$ is a finite and hence $A \cap B$ is a G_N−preopen set. □

Theorem 2.16. A subset A of a grill topological space (X,τ,G) is a G_N−preopen set if and only if for each $x \in A$ there is a G−preopen set U_x containing x and a finite subset F_x of X such that $U_x - F_x \subseteq A$.
Proof. Let A be a G_N−preopen set in a grill topological space (X,τ,G). Then for each $x \in A$ there is a G−preopen set U_x containing x and a such that $U_x - A$ is a finite set. Take $F_x = U_x - A$ and hence $U_x - F_x \subseteq A$.
Conversely, Let $x \in A$ be any point. Then there is a G−preopen set U_x containing x and a finite subset F_x of X such that $U_x - F_x \subseteq A$. Then $U_x - A \subseteq F_x$ and hence $U_x - A$ is a finite set. That is, A is a G_N−preopen set. □

Theorem 2.17. Let A be a G_N−preclosed set in a grill topological space (X,τ,G). Then $A \subseteq G \cup F$ for some G−preclosed set G and finite set F.
Proof. Since A is a G_N−preclosed set, then $X-A$ is a G_N−preopen set. Then by Theorem(2.16), for each $x \in X - A$ there is a G−preopen set U_x containing x and a finite subset F_x of X such that $U_x - F_x \subseteq X - A$. Hence

$$A \subseteq X - (U_x - F_x) = X - [U_x \cap (X - F_x)] = (X - U_x) \cup F_x.$$

Then $G = X - U_x$ is a G−preclosed set and $F = F_x$ is a finite set. □

Theorem 2.18. If (X,τ) is a T_1−space in a grill topological space (X,τ,G) then every nonempty G_N−preopen set contains nonempty G−preopen set.
Proof. Let A be any nonempty G_N−preopen set. Since A 6= ∅, let $x \in A$. Then there is a G−preopen set U_x containing x such that $U_x - A$ is a finite set. Since (X,τ) is a T_1−space, then by Theorem(1.1), $F = U_x - A$ is a closed set. Then $x \in U_x - F \subseteq A$ and $U_x - F = U_x \cap (X - F)$ by Lemma(2.14), $U_x - F$ is a nonempty G−preopen set. □

3 G_N−Preopen operators

In this section, we will define the interior operator and the closure operator via G_N−preopen sets.

Definition 3.1. Let (X,τ,G) be a grill topological space and $A \subseteq X$. The G_N−closure set of A is defined as the intersection of all G_N−preclosed subsets of X containing A and is denoted by ${}_{GN}Cl(A)$. The G_N−interior set of A is defined as the union of all G_N−preopen subsets of X contained in A and is denoted by ${}_{GN}Int(A)$.

Remark 3.2. From Theorem(2.9), ${}_{GN}Cl(A)$ is a G_N-preclosed set and ${}_{GN}Int(A)$ is G_N-preopen set in grill topological space (X,τ,G).

Remark 3.3. For a grill topological space (X,τ,G) and $A \subseteq X$, it is clear from the definition of ${}_{GN}Cl(A)$ and ${}_{GN}Int(A)$ that $A \subseteq {}_{GN}Cl(A)$ and ${}_{GN}Int(A) \subseteq A$.

Theorem 3.4. For a grill topological space (X,τ,G) and $A \subseteq X$, ${}_{GN}Cl(A) = A$ if and only if A is a G_N-preclosed set.
Proof. Let ${}_{GN}Cl(A) = A$. Then from definition of ${}_{GN}Cl(A)$ and Theorem(2.9), ${}_{GN}Cl(A)$ is a G_N-preclosed set and so A is a G_N-preclosed set. Conversely, we have $A \subseteq {}_{GN}Cl(A)$ by Remark(3.3). Since A is a G_N-preclosed set, then it is clear from the definition of ${}_{GN}Cl(A)$, ${}_{GN}Cl(A) \subseteq A$. Hence $A = {}_{GN}Cl(A)$. □

Theorem 3.5. For a grill topological space (X,τ,G) and $A \subseteq X$, ${}_{GN}Int(A) = A$ if and only if A is a G_N-preopen set.
Proof. Similar for proof of Theorem(3.4). □

Theorem 3.6. For a grill topological space (X,τ,G) and $A \subseteq X$, $x \in {}_{GN}Cl(A)$ if and only if for all G_N-preopen set U containing x, $U \cap A \neq \emptyset$.
Proof. Let $x \in {}_{GN}Cl(A)$ and U be a G_N-preopen set containing x. If $U \cap A = \emptyset$ then $A \subseteq X - U$. Since $X - U$ is a G_N-preclosed set containing A, then ${}_{GN}Cl(A) \subseteq X - U$ and so $x \in {}_{GN}Cl(A) \subseteq X-U$. Hence this is contradiction, because $x \in U$. Therefore $U \cap A \neq \emptyset$. Conversely, Let $x \notin {}_{GN}Cl(A)$. Then $X - {}_{GN}Cl(A)$ is a G_N-preopen set containing x. Hence by hypothesis, $[X - {}_{GN}Cl(A)] \cap A \neq \emptyset$. But this is contradiction, because $X - {}_{GN}Cl(A) \subseteq X - A$. □

Theorem 3.7. For a grill topological space (X,τ,G) and $A \subseteq X$, $x \in {}_{GN}Int(A)$ if and only if there is G_N-preopen set U such that $x \in U \subseteq A$.
Proof. Let $x \in {}_{GN}Int(A)$ and take $U = {}_{GN}Int(A)$. Then by Theorem(3.5) and definition of ${}_{GN}Int(A)$ we get that U is a G_N-preopen set and by Remark(3.3), $x \in U \subseteq A$. Conversely, Let there is G_N-preopen set U such that $x \in U \subseteq A$. Then by definition of ${}_{GN}Int(A)$, $x \in U \subseteq {}_{GN}Int(A)$. □

Theorem 3.8. For a grill topological space (X,τ,G) and $A,B \subseteq X$, the following hold:
1. If $A \subseteq B$ then ${}_{GN}Cl(A) \subseteq {}_{GN}Cl(B)$.
2. ${}_{GN}Cl(A) \cup {}_{GN}Cl(B) \subseteq {}_{GN}Cl(A \cup B)$.
3. ${}_{GN}Cl(A \cap B) \subseteq {}_{GN}Cl(A) \cap {}_{GN}Cl(B)$.
4. ${}_{GN}Cl(A) \subseteq Cl(A)$.

Proof. 1. Let $x \in {}_{GN}Cl(A)$. Then by Theorem(3.6), for all G_N-preopen set U containing x, $U \cap A \neq \emptyset$. Since $A \subseteq B$, then $U \cap B \neq \emptyset$. Hence $x \in {}_{GN}Cl(B)$. That is, ${}_{GN}Cl(A) \subseteq {}_{GN}Cl(B)$.
2. It is clear from the Part (1).
3. It is clear from the Part (1).
4. It is clear from Theorem(3.6) and from every open set U is G_N-preopen set. □

In the last theorem ${}_{GN}Cl(A \cup B) \neq {}_{GN}Cl(A) \cup {}_{GN}Cl(B)$ as it is shown in the following example.

Example 3.9. Let $(\mathbb{R},\tau_u,G)$ be a grill topological space on the set of real numbers $\mathbb{R}$ with usual topology τ_u and $G = P(\mathbb{R}) - \{\emptyset\}$. Let $A = IR - \{\sqrt{2}\}$ and $B = Q$, where Q is the set of rational numbers and IR is the set of irrational numbers. Since

$$Cl({}_{G}Int(A)) = Cl({}_{G}Int(IR - \{\sqrt{2}\})) = Cl(\emptyset) = \emptyset \subseteq A$$

and

$$Cl({}_{G}Int(B)) = Cl({}_{G}Int(Q)) = Cl(\emptyset) = \emptyset \subseteq B$$

then A and B are $G-$preclosed sets in R and hence are G_N-preclosed sets. Then

$${}_{GN}Cl(A) \cup {}_{GN}Cl(B) = A \cup B = \mathbb{R} - \{\sqrt{2}\}$$

If $\mathbb{R}-\{\sqrt{2}\}$ is a G_N-preclosed set then $\{\sqrt{2}\}$ is a G_N-preopen. Hence there is a $G-$preopen set U in R containing $\sqrt{2}$ such that $U-\{\sqrt{2}\}$ is a finite set. Then U is a finite set and hence

$$U \subseteq Int(\Psi(U)) = Int(U) = \emptyset$$

and this contradiction with $\sqrt{2} \in U$. Hence $\mathbb{R} - \{\sqrt{2}\}$ is not G_N-preclosed set in R. Since

$\mathbb{R} - \{\sqrt{2}\} \subseteq {}_{GN}Cl(\mathbb{R} - \{\sqrt{2}\})$ then

$${}_{GN}Cl(A \cup B) = {}_{GN}Cl(\mathbb{R} - \{\sqrt{2}\}) = \mathbb{R}.$$

Theorem 3.10. For a grill topological space (X,τ,G) and $A,B \subseteq X$, the following hold:
1. If $A \subseteq B$ then ${}_{GN}Int(A) \subseteq {}_{GN}Int(B)$.
2. ${}_{GN}Int(A) \cup {}_{GN}Int(B) \subseteq {}_{GN}Int(A \cup B)$.

3. $_{\mathcal{G}_N}Int(A \cap B) \subseteq {}_{\mathcal{G}_N}Int(A) \cap {}_{\mathcal{G}_N}Int(B)$.
4. $Int(A) \subseteq {}_{\mathcal{G}_N}Int(A)$.

Proof. Similar for Theorem(3.8). □

In the last theorem $_{\mathcal{G}_N}Int(A \cap B) \neq {}_{\mathcal{G}_N}Int(A) \cap {}_{\mathcal{G}_N}Int(B)$ as it is shown in the following example.

Example 3.11. Let $(\mathbb{R}, \tau_u, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with usual topology τ_u and $\mathcal{G} = P(\mathbb{R}) - \{\emptyset\}$. Let $A = Q \cup \{\sqrt{2}\}$ and $B = IR$. Since

$$A \subseteq Int(\Psi(A)) = Int(\Psi(Q \cup \{\sqrt{2}\})) = Int(\mathbb{R}) = \mathbb{R}$$

and

$$B \subseteq Int(\Psi(B)) = Int(\Psi(IR)) = Int(\mathbb{R}) = \mathbb{R}$$

then A and B are $\mathcal{G}$−preopen sets and hence are $\mathcal{G}_N$−preopen sets. Then

$$_{\mathcal{G}_N}Int(A) \cap {}_{\mathcal{G}_N}Int(B) = A \cap B = (Q \cap \{\sqrt{2}\}) \cap IR = \{\sqrt{2}\}$$

Since $\{\sqrt{2}\}$ is not $\mathcal{G}_N$−preopen and $_{\mathcal{G}_N}Int(\{\sqrt{2}\}) \subseteq \{\sqrt{2}\}$ then

$$_{\mathcal{G}_N}Int(A \cap B) = {}_{\mathcal{G}_N}Int(\{\sqrt{2}\}) = \emptyset.$$

Theorem 3.12. For a grill topological space $(X, \tau, \mathcal{G})$ and $A \subseteq X$, the following hold:
1. $_{\mathcal{G}_N}Int(X - A) = X - {}_{\mathcal{G}_N}Cl(A)$.
2. $_{\mathcal{G}_N}Cl(X - A) = X - {}_{\mathcal{G}_N}Int(A)$.

Proof. 1. Since $A \subseteq {}_{\mathcal{G}_N}Cl(A)$, then $X - {}_{\mathcal{G}_N}Cl(A) \subseteq X - A$. Since $_{\mathcal{G}_N}Cl(A)$ is a $\mathcal{G}_N$−preclosed set then $X - {}_{\mathcal{G}_N}Cl(A)$ is a $\mathcal{G}_N$−preopen set. Then

$$X - {}_{\mathcal{G}_N}Cl(A) = {}_{\mathcal{G}_N}Int[X - {}_{\mathcal{G}_N}Cl(A)] \subseteq {}_{\mathcal{G}_N}Int(X - A).$$

For the other side, let $x \in {}_{\mathcal{G}_N}Int(X - A)$. Then there is $\mathcal{G}_N$−preopen set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\mathcal{G}_N$−preclosed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathcal{G}_N}Cl(A)$, that is, $x \in X - {}_{\mathcal{G}_N}Cl(A)$.

2. Since $_{\mathcal{G}_N}Int(A) \subseteq A$, then $X - A \subseteq X - {}_{\mathcal{G}_N}Int(A)$. Since $_{\mathcal{G}_N}Int(A)$ is a $\mathcal{G}_N$−preopen set then $X - {}_{\mathcal{G}_N}Int(A)$ is a $\mathcal{G}_N$−preclosed set. Then

$$_{\mathcal{G}_N}Cl(X - A) = {}_{\mathcal{G}_N}Cl[X - {}_{\mathcal{G}_N}Int(A)] = X - {}_{\mathcal{G}_N}Int(A).$$

For the other side, let $x \notin {}_{\mathcal{G}_N}Cl(X - A)$. Then by Theorem(3.6), there is a $\mathcal{G}_N$−preopen set U containing x such that $U \cap (X - A) = \emptyset$. Then $x \in U \subseteq A$, that is, $x \in {}_{\mathcal{G}_N}Int(A)$. Hence $x \notin X - {}_{\mathcal{G}_N}Int(A)$. Therefore $X - {}_{\mathcal{G}_N}Int(A) \subseteq {}_{\mathcal{G}_N}Cl(X - A)$. □

Theorem 3.13. For a subset $A \subseteq X$ of grill topological space $(X, \tau, \mathcal{G})$, the following hold:
1. If G is an open set in X then $_{\mathcal{G}_N}Cl(A) \cap G \subseteq {}_{\mathcal{G}_N}Cl(A \cap G)$.
2. If G is a closed set in X then $_{\mathcal{G}_N}Int(A \cup G) \subseteq {}_{\mathcal{G}_N}Int(A) \cup G$.

Proof. (1) Let $x \in {}_{\mathcal{G}_N}Cl(A) \cap G$. Then $x \in {}_{\mathcal{G}_N}Cl(A)$ and $x \in G$. Let V be any $\mathcal{G}_N$−preopen set in $(X, \tau, \mathcal{G})$ containing x. By Theorem(2.15), $V \cap G$ is $\mathcal{G}_N$−preopen set containing x. Since $x \in {}_{\mathcal{G}_N}Cl(A)$ then by Theorem(3.6), $(V \cap G) \cap A \neq \emptyset$. This implies, $V \cap (G \cap A) \neq \emptyset$.

Hence by Theorem(3.6), $x \in {}_{\mathcal{G}_N}Cl(A \cap G)$. That is, $_{\mathcal{G}_N}Cl(A) \cap G \subseteq {}_{\mathcal{G}_N}Cl(A \cap G)$.

(2) Since G is a closed set X then by the part (1) and Theorem(3.12),

$$\begin{aligned} X - [{}_{\mathcal{G}_N}Int(A) \cup G] &= [X - {}_{\mathcal{G}_N}Int(A)] \cap [X - G] = [{}_{\mathcal{G}_N}Cl(X - A)] \cap [X - G] \\ &\subseteq {}_{\mathcal{G}_N}Cl[(X - A) \cap (X - G)] \\ &= {}_{\mathcal{G}_N}Cl(X - (A \cup G)) \\ &= X - ({}_{\mathcal{G}_N}Int(A \cup G)). \end{aligned}$$

Hence $_{\mathcal{G}_N}Int(A \cup G) \subseteq {}_{\mathcal{G}_N}Int(A) \cup G$. □

For a subset A of grill topological space $(X, \tau, \mathcal{G})$, the set

$$_{\mathcal{G}_N}\Gamma(A) = {}_{\mathcal{G}_N}Cl(A) - {}_{\mathcal{G}_N}Int(A)$$

is called $\mathcal{G}_N$−*frontier set* of A in $(X, \tau, \mathcal{G})$.

Theorem 3.14. For a subset $A \subseteq X$ of grill topological space $(X, \tau, \mathcal{G})$, the following hold:
1. $_{\mathcal{G}_N}Cl(A) = {}_{\mathcal{G}_N}\Gamma(A) \cup {}_{\mathcal{G}_N}Int(A)$.
2. $_{\mathcal{G}_N}\Gamma(A) \cap {}_{\mathcal{G}_N}Int(A) = \emptyset$.
3. $_{\mathcal{G}_N}\Gamma(A) = {}_{\mathcal{G}_N}Cl(A) \cap {}_{\mathcal{G}_N}Cl(X - A)$.

Proof. (1) Note that

$$\begin{aligned} {}_{\mathcal{G}_N}\Gamma(A) \cup {}_{\mathcal{G}_N}Int(A) &= ({}_{\mathcal{G}_N}Cl(A) - {}_{\mathcal{G}_N}Int(A)) \cup {}_{\mathcal{G}_N}Int(A) \\ &= [{}_{\mathcal{G}_N}Cl(A) \cap (X - {}_{\mathcal{G}_N}Int(A))] \cup {}_{\mathcal{G}_N}Int(A) \\ &= [{}_{\mathcal{G}_N}Cl(A) \cup {}_{\mathcal{G}_N}Int(A)] \cap [(X - {}_{\mathcal{G}_N}Int(A)) \cup {}_{\mathcal{G}_N}Int(A)] \\ &= {}_{\mathcal{G}_N}Cl(A) \cap X = {}_{\mathcal{G}_N}Cl(A). \end{aligned}$$

(2) It is clear from the definition of ${}_{GN}\Gamma(A)$.
(3) By Theorem(3.12),

$$\begin{aligned} \mathcal{G}_{\mathcal{N}}\Gamma(A) &= \mathcal{G}_{\mathcal{N}}Cl(A) - \mathcal{G}_{\mathcal{N}}Int(A) = \mathcal{G}_{\mathcal{N}}Cl(A) \cap (X - \mathcal{G}_{\mathcal{N}}Int(A)) \\ &= \mathcal{G}_{\mathcal{N}}Cl(A) \cap \mathcal{G}_{\mathcal{N}}Cl(X - A). \end{aligned}$$

□

Corollary 3.15. For a subset $A \subseteq X$ of grill topological space (X,τ,G), ${}_{GN}\Gamma(A)$ is G_N-preclosed set in (X,τ,G).
Proof. By Theorem(2.9) and the part (3) of the last theorem. □

Theorem 3.16. For a subset $A \subseteq X$ of grill topological space (X,τ,G), the following hold:
1. A is a G_N-preopen if and only if ${}_{GN}\Gamma(A) \cap A = \emptyset$.
2. A is a G_N-preclosed if and only if ${}_{GN}\Gamma(A) \subseteq A$.
3. A is both G_N-preopen and G_N-preclosed if and only if ${}_{GN}\Gamma(A) = \emptyset$.

Proof. (1) Let A be a G_N-preopen set. Then ${}_{GN}Int(A) = A$. Then by Theorem(3.14),

$${}_{GN}\Gamma(A) \cap A = {}_{GN}\Gamma(A) \cap {}_{GN}Int(A) = \emptyset.$$

Conversely, suppose that ${}_{GN}\Gamma(A) \cap A = \emptyset$. Then

$$\begin{aligned} A - {}_{GN}Int(A)) &= [A \cap {}_{GN}Cl(A)] - [A \cap {}_{GN}Int(A))] \\ &= A \cap ({}_{GN}Cl(A) - {}_{GN}Int(A)) = A \cap {}_{GN}\Gamma(A) = \emptyset. \end{aligned}$$

That is, ${}_{GN}Int(A) = A$. Hence A is a G_N-preopen set.
(2) Let A be a G_N-preclosed set. Then ${}_{GN}Cl(A) = A$. Then

$${}_{GN}\Gamma(A) = {}_{GN}Cl(A) - {}_{GN}Int(A) = A - {}_{GN}Int(A) \subseteq A.$$

Conversely, suppose that ${}_{GN}\Gamma(A) \subseteq A$. Then by Theorem(3.14),

$${}_{GN}Cl(A) = {}_{GN}Int(A)) \cup {}_{GN}\Gamma(A) \subseteq {}_{GN}Int(A)) \cup A \subseteq A.$$

That is, ${}_{GN}Cl(A) = A$. Hence A is G_N-preclosed set.

(3) Let A be both G_N-preclosed set and G_N-preopen set. Then ${}_{GN}Cl(A) = A = {}_{GN}Int(A)$. Then

$${}_{GN}\Gamma(A) = {}_{GN}Cl(A) - {}_{GN}Int(A) = A - A = \emptyset.$$

Conversely, suppose that ${}_{GN}\Gamma(A) = \emptyset$. Then ${}_{GN}Cl(A) - {}_{GN}Int(A) = \emptyset$. Since ${}_{GN}Int(A) \subseteq GN\ Cl(A)$ then ${}_{GN}Cl(A) = {}_{GN}Int(A)$. Since ${}_{GN}Int(A) \subseteq A \subseteq {}_{GN}Cl(A)$ then ${}_{GN}Cl(A) = A = {}_{GN}Int(A)$. That is, ${}_{GN}Cl(A) = A$. Hence A is both G_N-preclosed set and G_N-preopen set.

□

4 G_N-Relative and product spaces

We mean by *bitopological space* or *b-sapce* a triple (X,τ,ρ) consists two topologies τ and ρ on a set X. A subset $A \subseteq X$ is said to be *$\tau\rho-$preopen set* in a b-space (X,τ,ρ) if $A \subseteq {}_{\tau}Int({}_{\rho}Cl(A))$. The complement of $\tau\rho-$preopen set is said to be *$\tau\rho-$preclosed set*. A subset $A \subseteq X$ is said to be *$\tau\rho_N-$preopen set* in a b-space (X,τ,ρ) if for each $x \in A$, there exists a $\tau\rho-$preopen set U_x containing x such that U_x-A is a finite set. The complement of $\tau\rho_N-$preopen set is called $\tau\rho_N-$preclosed set.

Theorem 4.1. A subset $A \subseteq X$ is a $G-$preopen set in grill topological space (X,τ,G) if and only if it is a $\tau\tau_G-$preopen set in b-space (X,τ,τ_G).
Proof. It is clear from the definitions and $\Psi(A) = {}_{G}Cl(A)$. □

Theorem 4.2. A subset $A \subseteq X$ is a G_N-preopen set in grill topological space (X,τ,G) if and only if it is a $\tau\rho_N-$preopen set in b-space (X,τ,τ_G).
Proof. It is clear by Theorem(4.1). □

Theorem 4.3. A is $\tau\rho-$preclosed in b-space (X,τ,ρ) if and only if ${}_{\tau}Cl({}_{\rho}Int(A)) \subseteq A$.
Proof. A is a $\tau\rho-$preclosed set in X if and only if $X - A$ is a $\tau\rho-$preopen set in X if and only if

$$(X - A) \subseteq {}_{\tau}Int({}_{\rho}Cl(X - A)).$$

if and only if

$$\begin{aligned} (X - A) &\subseteq {}_{\tau}Int({}_{\rho}Cl(X - A)) = {}_{\tau}Int(X - {}_{\rho}Int(A)) \\ &= X - {}_{\tau}Cl({}_{\rho}Int(A)) \end{aligned}$$

if and only if ${}_{\tau}Cl({}_{\rho}Int(A)) \subseteq A$. □

Lemma 4.4. Let Y be an open subset of a grill topological space (X,τ,G). If A is a $G-$preopen set in (X,τ,G) then $A \cap Y$ is a $\tau|_Y\tau_G|_Y-$preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$.
Proof. Since A is $G-$preopen set in (X,τ,G) then $A \subseteq Int(\Psi(A))$. Then

$$\begin{aligned} A\cap Y &\subseteq Int(\Psi(A))\cap Y = Int(\Psi(A))\cap Int(Y) = Int(\Psi(A))\cap Y) \\ &= Int[(\Psi(A)\cap Y)\cap Y] \subseteq Int[\Psi(A\cap Y)\cap Y] = Int[{}_{\mathcal{G}}Cl(A\cap Y)\cap Y] \\ &= Int[{}_{\mathcal{G}}Cl|_Y(A\cap Y)] \subseteq Int|_Y[{}_{\mathcal{G}}Cl|_Y(A\cap Y)]. \end{aligned}$$

Hence $A \cap Y$ is a $\tau|_Y \tau_G|_Y$−preopen set in $(Y,\tau|_Y,\tau_G|_Y)$. □

Theorem 4.5. Let Y be an open subset of a grill topological space (X,τ,G). If A is a G_N−preopen set in (X,τ,G) then $A\cap Y$ is a $(\tau|_Y \tau_G|_Y)_N$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$.

Proof. Let A be a G_N−preopen set in (X,τ,G) and $x \in A \cap Y$. This implies $x \in A$ and $x \in Y$. Hence there is a G−preopen set U in X containing x such that $U - A$ is a finite. $x \in Y$ and by lemma above, the set $U \cap Y$ is a $\tau|_Y \tau_G|_Y$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ containing x and

$$\begin{aligned} (U\cap Y)\cap(Y-(A\cap Y)) &= (U\cap Y)\cap(Y\cap(X-A))) \\ &= U\cap(X-A)\cap Y \\ &= (U-A)\cap Y. \end{aligned}$$

Since $U-A$ is a finite, then $(U-A)\cap Y$ is a finite. That is, $A\cap Y$ is a$(\tau|_Y \tau_G|_Y)_N$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$. □

Corollary 4.6. Let Y be an open subset of a grill topological space (X,τ,G). If A is a G_N−preclosed set in (X,τ,G) then $A \cap Y$ is a $(\tau|_Y \tau_G|_Y)_N$−preclosed set in b-space $(Y,\tau|_Y,\tau_G|_Y)$.

Proof. Let A be a G_N−preclosed set in (X,τ,G). Then $X{-}A$ is a G_N−preopen set in (X,τ,G). By the last Theorem, $Y - A = (X - A) \cap Y$ is a $(\tau|_Y \tau_G|_Y)_N$−preopen set in $(Y,\tau|_Y,\tau_G|_Y)$.

Hence

$$Y - (Y - A) = Y - (Y \cap (X - A)) = Y \cap [(X - Y) \cup A] = A \cap Y$$

is a $(\tau|_Y \tau G|_Y)_N$−preclosed set in $(Y,\tau|_Y,\tau G|_Y)$. □

Lemma 4.7. Let Y be an open subset of a grill topological space (X,τ,G). If $A \subseteq Y$ is a $\tau|_Y \tau_G|_Y$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ then A G−preopen set in (X,τ,G).

Proof. Since A is $\tau|_Y \tau_G|_Y$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ then

$$\begin{aligned} A &\subseteq Int|_Y({}_{\mathcal{G}}Cl|_Y(A)) = Int({}_{\mathcal{G}}Cl|_Y(A)) \\ &= Int({}_{\mathcal{G}}Cl(A)\cap Y) \subseteq Int({}_{\mathcal{G}}Cl(A\cap Y)) \\ &= Int({}_{\mathcal{G}}Cl(A)) = Int(\Psi(A)) \end{aligned}$$

Hence A is a G−preopen set in (X,τ,G). □

Theorem 4.8. Let Y be an open subset of a grill topological space (X,τ,G). If $A \subseteq Y$ is a $(\tau|_Y \tau_G|_Y)_N$−preclosed set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ then A is G_N−preclosed set in (X,τ,G).

Proof. Let $x \in Y - A$. Since A is a $(\tau|_Y \tau_G|_Y)_N$−preclosed set in b-space $(Y,\tau|_Y,\tau_G|_Y)$, then there is a $\tau|_Y \tau_G|_Y$−preopen set U in $(Y,\tau|_Y,\tau_G|_Y)$ containing x such that $U \cap A = U \cap[Y-(Y-A)]$ is a finite. Since U is a $\tau|_Y \tau_G|_Y$−preopen in $(Y,\tau|_Y,\tau_G|_Y)$, then by lemma above, $U = O \cap Y$ for some G−preopen set O in X. Since Y is an open set in X and O is a preopen set in X, then $U = O \cap Y$ is G−preopen set in X containing x. Hence $Y - A$ is a G_N−preopen set in (X,τ,G), that is, A is a G_N−preclosed set in (X,τ,G). □

Corollary 4.9. Let Y be an open subset of a grill topological space (X,τ,G). If $A \subseteq Y$ is a $(\tau|_Y \tau_G|_Y)$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ then A is G_N−preopen set in (X,τ,G).

Theorem 4.10. Let Y be an open subset of a grill topological space (X,τ,G) and A be a subset of Y. Then ${}_{GN}Cl|_Y(A) = {}_{GN}Cl(A) \cap Y$.

Proof. Let $x \in {}_{GN}Cl|_Y(A)$ and G be a G_N−preopen set in (X,τ,G) containing x. By Theorem(4.5), $G \cap Y$ is a $(\tau|_Y \tau_G|_Y)_N$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ containing x and since $x \in {}_{GN}Cl|_Y(A)$, then

$$G \cap A = (G \cap Y) \cap A \neq \emptyset.$$

Hence $x \in {}_{GN}Cl(A)$, and since $x \in Y$, this implies $x \in {}_{GN}Cl(A) \cap Y$. That is,

$${}_{\mathcal{G}_N}Cl|_Y(A) \subseteq {}_{\mathcal{G}_N}Cl(A)\cap Y.$$

On the other side, let $x \in {}_{GN}Cl(A) \cap Y$ and O be a$(\tau|_Y \tau_G|_Y)_N$−preopen set in b-space $(Y,\tau|_Y,\tau_G|_Y)$ containing x. By Corollary (4.9), $O = G \cap Y$ for some G_N−preopen set G in (X,τ,G). Since $x \in {}_{GN}Cl(A)$, then $G \cap A$ 6= $\emptyset$ and so $(G \cap Y) \cap A$ 6= $\emptyset$, since $x \in Y$. Hence $O \cap A$ 6= $\emptyset$, that is, $x \in {}_{GN}Cl|_Y(A)$. Hence ${}_{GN}Cl(A) \cap Y \subseteq {}_{GN}Cl|_Y(A)$. □

Lemma 4.11. Let (X,τ,G) and (X^0,τ^0,G^0) be two grill topological spaces. Then $A \times B$ is $(\tau \times \tau^0)(\tau_G \times \tau_{G^0\,0})$−preopen set in b-space $(X \times X^0,\tau \times \tau^0,\tau_G \times \tau_{G^0\,0})$ if and only if A is a G−preopen set in (X,τ,G) and B is a G^0−preopen set in (X^0,τ^0,G^0).

Proof. Suppose that $A{\times}B$ is $(\tau{\times}\tau^0)(\tau_G{\times}\tau_{G^0\,0})$−preopen set in b-space $(X{\times}X^0,\tau{\times}\tau^0,\tau_G{\times}\tau_{G^0\,0})$.

Then

$$A \times B \subseteq {}_{\tau\times\tau}0Int[_{G\times G}0Cl(A \times B)].$$

Hence

$$\begin{aligned} A \times B &\subseteq {}_{\tau\times\tau}0Int[_{G\times G}0Cl(A \times B)] \\ &= {}_{\tau\times\tau}0Int[_{G}Cl(A) \times {}_{G}0Cl(B)] \\ &= {}_{\tau}Int(_{G}Cl(A)) \times {}_{\tau}0Int(_{G}0Cl(B)). \end{aligned}$$

This implies, $A \subseteq {}_{\tau}Int(_{G}Cl(A))$ and $B \subseteq {}_{\tau}0Int(_{G}0Cl(B))$. Then by Theorem(4.1), A is a G−preopen set in (X,τ,G) and B is a G^0−preopen set in (X^0,τ^0,G^0).

Conversely, suppose that A is a G−preopen set in (X,τ,G) and B is a G^0−preopen set in (X^0,τ^0,G^0). Then by Theorem(4.1), $A \subseteq {}_{\tau}Int(_{G}Cl(A))$ and $B \subseteq {}_{\tau}0Int(_{G}0Cl(B))$. Hence

$$\begin{aligned} A \times B &\subseteq {}_{\tau}Int(_{G}Cl(A)) \times {}_{\tau}0Int(_{G}0Cl(B)) \\ &= {}_{\tau\times\tau}0Int[_{G}Cl(A) \times {}_{G}0Cl(B)] \\ &= {}_{\tau\times\tau}0Int[_{G\times G}0Cl(A \times B)]. \end{aligned}$$

That is, $A \times B$ is $(\tau \times \tau^0)(\tau_G \times \tau_G{}^0{}_0)$−preopen set in b-space $(X \times X^0, \tau \times \tau^0, \tau_G \times \tau_G{}^0{}_0)$. □

Theorem 4.12. Let (X,τ,G) and $(X',\tau',\mathcal{G}')$ be two grill topological spaces. If $A \times B$ is a nonempty $(\tau \times \tau^0)(\tau_G \times \tau_G{}^0{}_0)_N$−preopen set in b-space $(X \times X^0, \tau \times \tau^0, \tau_G \times \tau_G{}^0{}_0)$ then A is a G_N−preopen set in (X,τ,G) and B is a $G_N{}^0$−preopen set in $(X',\tau',\mathcal{G}')$.

Proof. Let $x \in A$ be arbitrary point in A. Since $A \times B$ is a nonempty, take $y \in B$. Since$(x,y) \in A\times B$ and $A\times B$ is a $(\tau\times\tau')(\tau_{\mathcal{G}}\times\tau'_{\mathcal{G}'})_N-$preopen set in b-space $(X\times X', \tau\times\tau', \tau_{\mathcal{G}}\times\tau'_{\mathcal{G}'})$, then there is a $(\tau \times \tau')(\tau_{\mathcal{G}} \times \tau'_{\mathcal{G}'})-$preopen set $U \times G$ in b-space $(X \times X', \tau \times \tau', \tau_{\mathcal{G}} \times \tau'_{\mathcal{G}'})$containing (x,y) such that $(U \times G) - (A \times B)$ is a finite. Since

$$(U - A) \times (G - B) \subseteq (U \times G) - (A \times B),$$

then $(U-A)\times(G-B)$ is a finite, that is, $U-A$ is also a finite. Since $U\times G$ is a $(\tau\times\tau^0)(\tau_G \times \tau'_{\mathcal{G}'})-$preopen set $U\times G$ in b-space $(X \times X', \tau\times\tau', \tau_{\mathcal{G}}\times\tau'_{\mathcal{G}'})$ containing (x,y), then by lemmaabove U is a G−preopen set in (X,τ,G) containing x and G is a G−preopen set in $(X',\tau',\mathcal{G}')$containing y. Hence A is a G_N−preopen set in (X,τ,G). Similarly, B is a G_N−preopen set in $(X',\tau',\mathcal{G}')$. □

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