EPH - International Journal of Mathematics and Statistics
ISSN (Online): 2208-2212
Volume 7 Issue 2 March 2021

DOI:https://doi.org/10.53555/eijms.v7i1.53

COMPARISON BETWEEN ANALYSIS SOLUTIONS OF VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND

Abdel Radi Abdel Rahman Abdel Gadir[1*], Samia Abdallah Yagoub Ibrahim[2]
[*1]*Department of Mathematics, Faculty of Education Omdurman Islamic University, Omdurman, Sudan*
[2]*Researcher of Ph.D. in Mathematics in Omdurman Islamic University*

Corresponding Author:-

Abstract:-
Our goal in this paper is to compare and evaluate the accuracy and efficiency between the Volterra and Fredholm integral equations of the second kind with initial condition. We followed the Adomian Decomposition method and series solution method, and we found that the two methods in terms of accuracy in the solution also we found that the Adomian Decomposition method gave us the more accurate solution than the other method, so the Adomian Decomposition method it's the best one method.

Key Words:-*Volterra and Fredholm integral equations, Adomian decomposition method series solution method.*

1. INTRODUCTION

The emergence of the theory of integral equations was the need for mathematicians to address some of the problems in mathematical engineering and the problems of vibrations in mechanics. He was the first to write in this field a veto Volterra in the late 19th century AD where he laid the basic concepts of this theory but he had no way to solve. This paved the way for Fredholm in 1900 to give a solution to these integral equations, especially nonlinear ones.The Volterra-Fredholm integral equations ([12] and [5]) arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)u(t)\,dt \tag{1}$$

and the mixed form

$$u(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)u(t)\,dt\,dr \tag{2}$$

where $f(x)$ and $K(x,t)$ are analytic functions. It is interesting to note that (1) contains disjoint Volterra and Fredholm integrals, whereas (2) contains mixed Volterra and Fredholm integrals. Moreover, the unknown functions $u(x)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Examples of the two types of the Volterra-Fredholm integral equations of the second kind are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)\,dt - \int_0^1 tu(t)\,dt$$

$$u(x) = x + \frac{17}{2}x^2 - \int_0^x \int_0^1 (r-t)u(t)\,dr\,dt$$ (3) and (4)

2.Volterra Integral Equations of Second Kind

We will first study Volterra integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \tag{5}$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods, new and traditional, that will be used.

2.1 The Adomian Decomposition Method

The Adomian decomposition method appears to work for linear and nonlinear differential equations, integral equations, integro-differential equations. The method was introduced by Adomian in early 1990 in his books [1] and [2] and other related research papers [3] and [4] The method essentially is a power series method similar to the perturbation technique. We shall demonstrate the method by expressing $u(x)$ in the form of a series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{6}$$

with $u_0(x)$ as the term outside the integral sign.
The integral equation is

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \tag{7}$$

and hence

$$u_0(x) = f(x) \tag{8}$$

Substituting equation (6) into equation (7) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left\{ \sum_{n=0}^{\infty} u_n(x) \right\} dt \tag{9}$$

The components $u_0(x), u_1(x), u_2(x), \cdots, u_n(x), \cdots$ of the unknown function $u(x)$ can be completely determined in a recurrence manner if we set

$$
\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \lambda \int_0^x K(x,t)u_0(t)\,dt \\
u_2(x) &= \lambda \int_0^x K(x,t)u_1(t)\,dt \\
\cdots &= \cdots \\
u_n(x) &= \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt
\end{aligned} \tag{10}
$$

and so on. This set of equations (10) can be written in compact recurrence scheme as

$$
u_0(x) = f(x)
$$

$$
u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)\,dt,\ n \geq 0 \tag{11}
$$

It is worth noting here that it may not be possible to integrate the kernel for many components. In that case, we truncate the series at a certain point to approximate the function $u(x)$. There is another point that needs to be addressed; that is the convergence of the solution of the infinite series. This problem was addressed by many previous workers in this area (see Refs. [10] and [9]). So, it will not be repeated here. We shall demonstrate the technique with an example.

Example 2.1. *Solve the following Volterra integral equation:*

$$
u(x) = 1 + \int_0^x (t-x)u(t)\,dt \tag{12}
$$

Solution:

We notice that $f(x) = 1, \lambda = 1, K(x,t) = t - x$. Substituting the decomposition series (6) into both sides of (12) gives

$$
\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} (t-x)u_n(t)\,dt \tag{13}
$$

or equivalently

$$
u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x (t-x)[u_0(x) + u_1(x) + u_2(x) + \cdots]\,dt \tag{14}
$$

that gives

$$
\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= \int_0^x (t-x)u_0(t)\,dt = \int_0^x (t-x)\,dt = -\frac{1}{2!}x^2 \\
u_2(x) &= \int_0^x (t-x)u_1(t)\,dt = -\frac{1}{2!}\int_0^x (t-x)t^2\,dt = \frac{1}{4!}x^4 \\
u_3(x) &= \int_0^x (t-x)u_2(t)\,dt = \frac{1}{4!}\int_0^x (t-x)t^4\,dt = -\frac{1}{6!}x^6 \\
u_4(x) &= \int_0^x (t-x)u_3(t)\,dt = -\frac{1}{6!}\int_0^x (t-x)t^6\,dt = \frac{1}{8!}x^8
\end{aligned} \tag{15) and (16}
$$

and so on. The solution in a series form is given by

$$
u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \tag{17}
$$

and in a closed form by

$$
u(x) = \cos x \tag{18}
$$

obtained upon using the Taylor expansion for $\cos x$.

2.2 The Series Solution Method

We shall introduce a practical method to handle the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \tag{19}$$

In the series solution method we shall follow a parallel approach known as the Frobenius series solution usually applied in solving the ordinary differential equation around an ordinary point (see Ref. [13] and [15]). The method is applicable provided that $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylors expansion around $x = 0$. Accordingly,$u(x)$ can be expressed by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{20}$$

where the coefficients a and x are constants that are required to be determined. Substitution of equation (20) into the above Volterra equation yields

$$\sum_{n=0}^{\infty} a_n x^n = f(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{\infty} a_n t^n\,dt \tag{21}$$

so that using a few terms of the expansion in both sides, we find

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ &= f(x) + \lambda \int_0^x K(x,t) a_0\,dt + \lambda \int_0^x K(x,t) a_1 t\,dt \\ &\quad + \lambda \int_0^x K(x,t) a_2 t^2\,dt + \cdots + \lambda \int_0^x K(x,t) a_n t^n\,dt + \cdots \end{aligned} \tag{22}$$

In view of equation (22), the integral equation will be reduced to several traditional integrals, with defined integrals having terms of the form $t^n, n \geq 0$ only. We then write the Taylors expansions for $f(x)$ and evaluate the first few integrals in equation (22). Having performed the integration, we equate the coefficients of like powers of x in both sides of equation (22). This will lead to a complete determination of the unknown coefficients $a_0, a_1, a_2, \cdots, a_n, \cdots$ Consequently, substituting these coefficients $a_n, n \geq 0$, which are determined in equation (22), produces the solution in a series form. We will illustrate the series solution method by a simple example.

Example 2.2. *Solve the Volterra integral equation by using the series solution method*

$$u(x) = 1 + \int_0^x u(t)\,dt \tag{23}$$

Solution:

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{24}$$

into both sides of Eq. (23) leads to

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt \tag{25}$$

Evaluating the integral at the right side gives

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1} \tag{26}$$

that can be rewritten as

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n \tag{27}$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots. \tag{28}$$

In (26), the powers of x of both sides are different, therefore, we make them the same by changing the index of the second sum to obtain (27). Equating the coefficients of like powers of x in both sides of (27) gives the recurrence relation

$$a_0 = 1$$

$$a_n = \frac{1}{n} a_{n-1}, \quad n \geq 1 \tag{29}$$

where this result gives

$$a_n = \frac{1}{n!}, \quad n \geq 0 \tag{30}$$

Substituting this result into (24) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{31}$$

that converges to the exact solution $u(x) = e^x$.

It is interesting to point out that this result can be obtained by equating coefficients of like terms in both sides of (28), where we find

$$\begin{aligned} a_0 &= 1 \\ a_0 &= a_1 = 1 \\ &\vdots \\ a_n &= \frac{1}{n} a_{n-1} = \frac{1}{n\,!} \end{aligned} \tag{32}$$

This leads to the same result obtained before by solving the recurrence relation.

3. Fredholm Integral Equations of the Second Kind

We will first study Fredholm integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t)\, dt \tag{33}$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods, new and traditional, that will be used to handle the Fredholm integral equations (33).

3.1The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in [1], [3], [6] and [7]. The Adomian method will be briefly outlined. The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{34}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{35}$$

where the components $u_n(x), n \geq 0$ will be determined recurrently. The Adomian decomposition method concerns itself with finding the components $u_0, u_1, u_2, \cdots$ individually. As we have seen before, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (34) into the Fredholm integral equation (33) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \tag{36}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_a^b K(x,t) \left[u_0(t) + u_1(t) + u_2(t) + \cdots \right] dt \tag{37}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. This means that the components $u_j(x), j \geq 0$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation

$$u_0(x) = f(x), \quad u_{n+1} = \lambda \int_a^b K(x,t) u_n(t)\, dt, \; n \geq 0 \tag{38}$$

or equivalently

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= f(x) + \lambda \int_a^b K(x,t) u_0(t)\, dt \\ u_2(x) &= f(x) + \lambda \int_a^b K(x,t) u_1(t)\, dt \\ u_3(x) &= f(x) + \lambda \int_a^b K(x,t) u_2(t)\, dt \end{aligned} \tag{39}$$

and so on for other components.

In view of (39), the components $u_0(x), u_1(x), u_2(x), u_3(x), \cdots$ are completely determined. As a result, the solution $u(x)$ of the Fredholm integral equation (33) is readily obtained in a series form by using the series assumption in (34).
It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown that if an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 3.1. *Solve the following Fredholm integral equation*

$$u(x) = 2 + \cos x + \int_0^\pi t u(t)\, dt \tag{40}$$

Solution:
Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x) = 2 + \cos x + \int_0^\pi t \sum_{n=0}^{\infty} u_n(t)\, dt \tag{41}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 2 + \cos x + \int_0^\pi t \sum_{n=0}^{\infty} [u_0(t) + u_1(t) + u_2(t) + \cdots]\, dt \tag{42}$$

We next set the following recurrence relation

$$u_0(x) = 2 + \cos x, \; u_{k+1}(x) = \int_0^\pi t u_k(t)\, dt, \; k \geq 0 \tag{43}$$

This in turn gives

$$\begin{aligned} u_0(x) &= 2 + \cos x \\ u_1(x) &= \int_0^\pi u_0(t)\, dt = -2 + \pi^2 \\ u_2(x) &= \int_0^\pi u_1(t)\, dt = -\pi^2 + \frac{1}{2}\pi^4 \\ u_3(x) &= \int_0^\pi u_2(t)\, dt = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^6 \\ u_4(x) &= \int_0^\pi u_3(t)\, dt = -\frac{1}{4}\pi^6 + \frac{1}{8}\pi^8 \end{aligned} \tag{44}$$

and so on. Using (34) gives the series solution

$$\begin{aligned} u_0(x) = 2 + \cos x &+ (-2 + \pi^2) + (-\pi^2 + \frac{1}{2}\pi^4) \\ &+ (-\frac{1}{2}\pi^4 + \frac{1}{4}\pi^6) + (-\frac{1}{4}\pi^6 + \frac{1}{8}\pi^8) + \cdots \end{aligned} \tag{45}$$

We can easily observe the appearance of the noise terms, i.e the identical terms with opposite signs. Canceling these noise terms in (45) gives the exact solution

$$u(x) = \cos x \tag{46}$$

3.2The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{k} \frac{u^{(n)}(b)}{n!}(x-b)^n \tag{47}$$

converges to $f(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{48}$$

The series solution method that stems mainly from the Taylor series for analytic functions, will be used for solving Fredholm integral equations. We will assume that the solution $u(x)$ of the Fredholm integral equations

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt \tag{49}$$

is analytic, and therefore possesses a Taylor series of the form given in (48), where the coefficients a_n will be determined recurrently. Substituting (48) into both sides of (49) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \tag{50}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda \int_a^b K(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt \tag{51}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (49) will be converted to a traditional integral in (50) or (51) where instead of integrating the unknown function $u(x)$, terms of the form $t^n, n \geq 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (50) or (51), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geq 0$. Having determined the coefficients $a_j, j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (48). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

It is worth noting that using the series solution method for solving Fredholm integral equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x, e^x$, etc, the series method gives the exact solution after rounding few of the coefficients $a_j, j \geq 0$. This will be illustrated by studying the following example.

Example 3.2. *Solve the Fredholm integral equation by using the series solution method*

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t)\,dt \tag{52}$$

Solution:

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{53}$$

into both sides of Eq. (52) gives

$$\sum_{n=0}^{\infty} a_n x^n = -1 + \cos x + \int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} a_n t^n\right) dt \tag{54}$$

Evaluating the integral at the right side, using the Taylor series of $\cos x$, and proceeding as before we find

$$a_0 = 1,\ a_{2j+1} = 0,\ a_{2j} = \frac{(-1)^j}{(2j)!},\ j \geq 0 \tag{55}$$

Consequently, the exact solution is given by

$$u(x) = \cos x \tag{56}$$

4. The Volterra and Fredholm Integral Equations

In this part, we will study some of the reliable methods that will be used for analytic treatment of the Volterra-Fredholm integral equations ([12] and [5]) of the form

$$u(x) = f(x) + \int_0^x K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u(t)\,dt \tag{57}$$

4.1 The Adomian Decomposition Method

The Adomian decomposition method [1], [2], [8] and [6] (ADM) was introduced thoroughly in this text for handling independently Volterra-Fredholm integral equations. The method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{58}$$

where the components $u_n(x), n \geq 0$ are to be determined in a recursive manner. To establish the recurrence relation, we substitute the decomposition series into the Volterra-Fredholm integral equation (57) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x K_1(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt + \int_a^b K_2(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \tag{59}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, we set the recurrence relation

$u_0 = f(x)$

$$u_{n+1}(x) = \int_0^x K_1(x,t)u_n(t)dt + \int_a^b K_2(x,t)u_n(t)dt \quad n \geq 0 \tag{60}$$

Having determined the components $u_0(x), u_1(x), u_2(x), \cdots$, the solution in a series form is readily obtained upon using (58). The series solution converges to the exact solution if such a solution exists. This will be illustrated by using the following example.

Example 4.1. *Use the Adomian decomposition method to solve the following VolterraFredholm integral equation*

$$e^x + 1 + x + \int_0^x (x-t)u(t)dt - \int_0^1 e^{x-t}u(t)dt \tag{61}$$

Solution:

Using the decomposition series (58), and using the recurrence relation (60) we obtain

$$\begin{aligned} u_0(x) &= e^x + 1 + x \\ u_1(x) &= \int_0^x (x-t)u(t)\,dt - \int_0^1 e^{x-t}u(t)\,dt \\ &= -x - 1 + \frac{1}{2}x^2 + \cdots \end{aligned} \tag{62}$$

and so on. We notice the appearance of the noise terms ± 1 and $\pm x$ between the components $u_0(x)$ and $u_1(x)$. By canceling these noise terms from $u_0(x)$, the non-canceled term of $u_0(x)$ gives the exact solution

$$u(x) = e^x \tag{63}$$

that satisfies the given equation.

4.2 The Series Solution Method

The series solution method was examined before in this text . A real function $u(x)$ is called analytic if it has derivatives of all orders such that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{64}$$

In this part we will apply the series solution method ([11] and [14]), that stems mainly from the Taylor series for analytic functions, for solving Volterra-Fredholm integral equations. We will assume that the solution $u(x)$ of the Volterra-Fredholm integral equation

$$u(x) = f(x) + \int_0^x K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u(t)\,dt \tag{65}$$

is analytic, and therefore possesses a Taylor series of the form given in (64), where the coefficients a_n will be determined recurrently. In this method, we usually substitute the
Taylor series (64) into both sides of (65) to obtain

$$\sum_{k=0}^{\infty} a_k x^k = f(x) + \int_0^x K_1(x,t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt + \int_a^b K_2(x,t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt \tag{66}$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \int_0^x K_1(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt + \int_a^b K_2(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) dt \tag{67}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The Volterra-Fredholm integral equation (65) will be converted to a regular integral in (66) or (67) where instead of integrating the unknown function $u(x)$, terms of the form $t^n, n \geq 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.
We first integrate the right side of the integrals in (66) or (67), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geq 0$. Having determined the coefficients $a_j, j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (64). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.

Example 4.2. *Solve the Volterra-Fredholm integral equation by using the series solution method*

$$u(x) = e^x - 1 - x + \int_0^x u(t)\,dt + \int_0^1 x u(t)\,dt \tag{68}$$

Solution:
Using the Taylor polynomial for e^x up to x^7, substituting $u(x)$ by the Taylor polynomial

$$u(x) = \sum_{n=0}^{7} a_n x^n \tag{69}$$

and proceeding as before leads to

$$\sum_{n=0}^{7} a_n x^n = \left(2a_0 + \sum_{n=0}^{7} \frac{1}{n+1} a_n\right) x + \frac{1+a_1}{2!}x^2 + \frac{1+2!a_2}{3!}x^3 + \frac{1+3!a_3}{4!}x^4 + \frac{1+4!a_4}{5!}x^5 + \frac{1+5!a_5}{6!}x^6 + \frac{1+6!a_6}{7!}x^7 + \cdots \tag{70}$$

Equating the coefficients of like powers of x in both sides of (70), and proceeding as before we obtain

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = \tfrac{1}{2!}$$
$$a_4 = \tfrac{1}{3!}, \quad a_5 = \tfrac{1}{4!}, \quad a_6 = \tfrac{1}{5!} \tag{71}$$

The exact solution is given by

$$u(x) = xe^x \tag{72}$$

Results

After we solved the Volterra and Fredholm integral equations of the second kind with initial condition, we compared between the two solutions and we found that: The Adomian Decomposition method is straightforward to implement in the analysis test cases considered leads to very significant improvement in accuracy and also (ADM) has main advantages such as simplicity, high accuracy and if the solution it exists it found in a rapidly convergent series form.

Conclusion

In this paper we solved and compared between two analysis method of Volterra and Fredholm integral equation of the second kind using the (ADM) and the (SSM). In most contemporary studies involving Volterra and Fredholm integral equations some projection methods are being developed with the old of interpolation projections and approximation theory here we considered the (ADM) for these equations and explained that this method is straightforward to implement and in the analysis test cases considered leads to very significant improvement in accuracy also we gave a bound for (ADM) series and the efficiency and effectiveness.

References

[1]. Adomian, G., Solving Frontier Problems of Physics, the Decomposition Method, *Kluwer Boston,* 1994.
[2]. Adomian, G., Nonlinear Stochastic Operator Equations, *Academic Press: San Diego, CA 1986*.
[3]. Adomian, G., A Review of the Decomposition Method and Some Recent Results for Nonlinear Equation,*Math. Comput. Modelling*, 13(7), pp. 17-43, 1992.
[4]. Adomian, G. & Rach, R., Noise Terms in Decomposition Series Solution, *Computers Math. Appl.*, 24 (1992) 61-64.
[5]. A.M. Wazwaz, A Reliable Treatment for Mixed Volterra-Fredholm Integral Equations, *Appl. Math. Comput.* 127 (2002) 405-414.
[6]. A.M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, *HEP and Springer Beijing and Berlin*, 2009.
[7]. A.M. Wazwaz, a First Course in Integral Equations, *World Scientific Singapore*, (1997).
[8]. A.M.Wazwaz, A Reliable Modification of Adomian Decomposition Method,*Appl. Math. Comput.*, 92 (1998) 1-7.
[9]. Cherruault, Y. & Adomian, G., Decomposition methods: A new proof of convergence,*Mathl. Comput. Modelling*, 18(12), pp. 103-106, 1993.
[10]. Cherruault, Y., Saccomandi, G. & Some, B., New Results for Convergence of Adomian's Method Applied to Integral Equations,*Mathl. Comput. Modelling* 16(2), pp. 85-93, 1993.
[11]. H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, *Dover, New York* 1962.
[12]. K. Maleknejad and M. Hadizadeh, a New Computational Method for Volterra-Fredholm Integral Equations, *Comput. Math. Appl.*, 37 (1999), 1-8.
[13]. Lighthill, M.J., Fourier analysis and Generaralised Functions, *Cambridge University Press: Cambridge*, 1964.
[14]. R. Kress, Linear Integral Equations, *Springer, Berlin*, 1999.
[15]. Rahman, M.,Mathematical Methods with Applications, *WIT Press: Southampton, UK,* 2000.