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STATISTICAL ANALYSIS OF GEOMETRIC DISTRIBUTION UNDER CONSTANT-STRESS ACCELERATED LIFE TEST

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Abstract:-
Firstly, this paper gives the maximum likelihood estimation of the parameter of geometric distribution under fix-group and fix-time censored test, then sets a linear and regression model using the asymptotic normality of MLE and makes statistical analysis of geometric distribution under constant-stress accelerated life test.

Keywords:- *Geometric distribution; censoring life test; maximum likelihood estimation; constant-stress accelerated life test; least square method*

Mathematics Subject Classification: 62N05

1. INTRODUCTION

The life of many products is discrete, such as some connector products (such as switches, etc.), the life can be described by geometric distribution. Geometric distribution plays a very important role in reliability theory and applied probabilistic model because of its memorylessness. In the discrete life case, geometric distribution plays the role of exponential distribution in the continuous case, so the study of geometric distribution products has important theoretical and application value. Although the statistical analysis of geometrically distributed products is not as mature as exponential distribution, there are some research results, which can be found in references [1–10]. In this paper, for the first time, we propose a geometric distribution of the fixed group truncation life test, even the timed truncation can be reduced to the fixed group truncation.

This paper gives the maximum likelihood estimation of the parameter of geometric distribution under fix-group and fix-time censored test, then sets a linear and regression model using the asymptotic normality of MLE and makes statistical analysis of geometric distribution under constant-stress accelerated life test.

The rest of this paper is organized as follows. In Section 2, we obtain the MLE of geometric distribution parameter under censoring life test with certain stress conditions. Section 3 makes statistical analysis of geometric distribution under constant-stress accelerated life test. First of all, it gives the arrangement and basic assumptions of tests, then makes the statistical analysis under censored sample. In Section 4, we make the statistical analysis of the data from a specific example using the previous statistical methods. Finally, we summarize and conclude the paper in Section 5.

2. MLE of geometric distribution parameter under censoring life test with certain stress conditions

When the product life follows the continuous distribution, the fixed and timed censoring life tests are often used in the product life tests. If the product life follows the discrete distribution, the fixed group censoring life test and the fixed timed censoring life test can be used when the product life test is carried out. The fixed group censoring life test is to stop the test when r groups of different failure data are observed.

Suppose the product life T follows geometric distribution $Geo(p)$, and its probability distribution is

$$P(T=t)=pq^{t-1},\ t=1,2,3,\cdots;\ 0<p<1, q=1-p.$$

Randomly selected n products were subjected to a fixed group of censoring life test under a certain stress condition, or all were subjected to a fix-time censoring life test. There are r groups of failures in the n products, and the failure data is

$$1\le t_1<t_2<\cdots<t_r\le\tau,$$

where the failure data of m_i of the n products is t_i ($i=1,2,\cdots,r$), and τ is the time at which the test is to be stopped in advance. In a fixed group censoring life test, $\tau=t_r$. In order to give the maximum likelihood estimation of p, the likelihood function of the sample can be written as

$$\begin{aligned}L(p)&=\frac{n!}{m_1!\cdots m_r!m_{r+1}}[\prod_{i=1}^{r}(pq^{t_i-1})^{m_i}](q^{\tau})^{n-m_1-\cdots-m_r}\\&=\frac{n!}{m_1!\cdots m_r!m_{r+1}}p^{\sum\limits_{i=1}^{r}m_i}q^{\sum\limits_{i=1}^{r}m_it_i+m_{r+1}\tau-\sum\limits_{i=1}^{r}m_i}\\&=\frac{n!}{m_1!\cdots m_r!m_{r+1}}p^{M_r}q^{T_r-M_r},\quad 1\le k_1<k_2<\cdots<k_r\end{aligned},$$

Where $m_1+m_2+\cdots+m_{r+1}=n, T_r=\sum_{i=1}^{r}m_it_i+m_{r+1}\tau$ is called as total test time, and $M_r=\sum_{i=1}^{r}m_i$ is the number of failures.

The logarithmic likelihood function is

$$\ln L(p)=k+M_r\ln p+(T_r-M_r)\ln(1-p),$$

where $k=\ln\frac{n!}{m_1!\cdots m_r!m_{r+1}}$ is a constant independent of the parameter p.

Set

$$\frac{\partial\ln L(p)}{\partial p}=0,$$

the maximum likelihood estimation (MLE) of p is obtained as

$$\hat{p}=\frac{M_r}{T_r}.$$

3. Statistical analysis of geometric distribution under constantstress accelerated life test

3.1. Arrangement and basic assumptions of tests

The arrangement of constant-stress accelerated life test is as follows:

(1) Determine the normal stress level S_0 and k accelerated stress levels $S_1,\cdots,S_k$, these stress levels should generally satisfy the following relationships:

$$S_0<S_1<\cdots<S_k.$$

(2) The n products are randomly selected from a batch of products and divided into k samples with sample sizes of $n_1,\cdots,n_k$ respectively, and $n_1+\cdots+n_k=n$. The ith sample will be arranged under S_i for life test.

(3) At k accelerated stress levels, the fix-group censored tests are carried out, or the fix-time censored tests are carried out. In n_i products under S_i, there are r_i groups of failures, and the failure data is

$$1\ 1 \le t_{i1} < t_{i2} < \cdots < t_{ir_i} \le \tau_i\ i = 1,2,\cdots,k,$$

where the failure data of m_{ij} of the n_i products is t_{ij} $(j = 1,2,\cdots,r_i)$, τ_i is the censoring time of the product under S_i. If the censored test is fix-time, then $\tau i = tiri$.

The statistical analysis of geometric distribution under constant-stress accelerated life test is carried out under the following two basic assumptions:

(1) Under normal stress level S_0 and accelerated stress levels $S_0 < S_1 < \cdots < S_k$ product life T_i $(i = 0,1,\cdots,k)$ all follow geometric distributions, whose probability distribution are respectively

$$P(T_i = t) = p_i(1-p_i)^{t-1},\ t = 1,2,\cdots,\ i = 0,1,2,\cdots,k.$$

Set $\theta_i = p_i^{-1}$, easy to know θ_i and p_i are the average life and failure efficiency of products under S_i respectively.

(2) The following acceleration model is satisfied between the average life θ_i of the product and the level S_i of acceleration stress used:

$$\ln\theta_i = a + b\phi(S_i),\ i = 1,2,\cdots,k,$$

where a and b are the parameters to be estimated, and $\phi(S)$ is the known function of S (when the stress level S is the absolute temperature, then $\phi(S) = 1/S$, and the acceleration model is called Arrhenius model; when the stress level S is the voltage, then $\phi(S) = \ln S$, and the acceleration model is called the inverse power law model).

Under the above arrangements and assumptions, the statistical analysis of constant-stress accelerated test data is carried out in the following three steps:

(1) Try to get the point estimation of parameters a and b in the acceleration model;

(2) According to assumption (2), the point estimate of average life θ_0 under normal stress level S_0 is obtained;

(3) According to assumption (1), The probability distribution of geometric distribution under S_0 is obtained, thus it is not difficult to calculate the estimation of various reliability indexes under S_0.

3.2. Statistical analysis under censored sample

In the constant-stress accelerated life test, the total test time of censored test under the accelerated stress level S_i is

$$T_i^* = \sum_{j=1}^{r_i} m_{ij}t_{ij} + (n - m_{i1} - \cdots - m_{ir_i})\tau_i\ ,\quad i = 1,2,\cdots,k.$$

Although the distribution of T_i^* is difficult to obtain, the asymptotic normality of maximum likelihood estimation can be used for statistical analysis. However, the sample size and failure number are both large at each stress level. Let's describe this method in detail.

(1) According to the assumption (1), the life distribution under the accelerated stress level S_i is geometric, so the MLE of its average life θ_i is

$$\hat{\theta}_i = 1/\hat{p}_i = T_i^* / \sum_{j=1}^{r_i} m_{ij}\ ,\quad i = 1,2,\cdots,k. \tag{1}$$

(2) According to the assumption (1) and asymptotic normality of MLE, the following one-variable linear regression model can be established

$$\ln\hat{\theta}_i = a + b\phi(S_i) + \varepsilon_i,\qquad i = 1,2,\cdots,k,$$

where ε_i is random error term, $E(\varepsilon_i) = 0, Var(\varepsilon_i) = \sigma^2$, and $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k$ are independent of each other.

(3) The least square estimation (LSE) of a and b can be obtained as by the least square method for the above one-variable linear regression

$$\hat{a} = \bar{y} - \hat{b}\bar{x}\ ,\ \hat{b} = l_{xy}/l_{xx}\ ,$$

where

$$\bar{y} = \frac{1}{k}\sum_{i=1}^{k} y_i,\qquad y_i = \ln\hat{\theta}_i,$$

$$\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i,\qquad x_i = \varphi(S_i),$$

$$l_{xx} = \sum_{i=1}^{k}(x_i-\bar{x})^2 = \sum_{i=1}^{k} x_i{}^2 - \frac{1}{k}(\sum_{i=1}^{k} x_i)^2,$$

$$l_{xy} = \sum_{i=1}^{k}(x_i-\bar{x})(y_i-\bar{y}) = \sum_{i=1}^{k} x_iy_i - \frac{1}{k}(\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} y_i).$$

This leads to the following acceleration model

$$\ln \hat{\theta}_i = \hat{a} + \hat{b}\varphi(S_i)\ , \quad i = 0, 1, 2, \cdots, k.$$

(4) Point estimation of reliability index.

If $S = S_0$, the point estimate of $\ln\hat{\theta}_0$ can be obtained from the acceleration equation as follows:

$$\ln\hat{\theta}_0 = \hat{a} + \hat{b}\phi(S_0).$$

From above, the estimation of average life under normal stress level S_0 can be obtained as $\hat{\theta}_0 = e^{\hat{a}+\hat{b}\phi(S0)}$, then the estimation of p_0 (namely failure efficiency) can be obtained as $\hat{p}_0 = e^{-[\hat{a}+\hat{b}\phi(S0)]}$, and finally according to $\hat{p}_0$, the estimation of other reliability indexes can be obtained. For example, the estimation of reliability $R(t) = P(T_0 > t) = (1-p)t$ is $\hat{R}(t) = \{1 - e{-}[\hat{a}+\hat{b}\phi(S_0)]\}^t$.

To obtain the estimation of the acceleration coefficient $\tau_{Si\sim S0}$, the expression of the acceleration coefficient for α quantile lifetime is first required. Suppose the failure distribution function of the product under the stress level S_i is $F_i(t)$ $(i = 0,1,2,\cdots,k)$, then the acceleration coefficient of α quantile lifetime is $\tau_{Si\sim S0} = \frac{t_{\alpha,0}}{t_{\alpha,i}}$, where $F_0(t_{\alpha,0}) = \alpha, F_i(t_{\alpha,i}) = \alpha$, it is easy to derive $\tau_{S_i\sim S_0} = \frac{\ln(1-p_i)}{\ln(1-p_0)}$, and the acceleration coefficient is independent of α. So the estimate of the acceleration coefficient is $\hat{\tau}_{S_i\sim S_0} = \frac{\ln \hat{}}{\ln \hat{q}_0} = \frac{\ln(1-\hat{p}_i)}{\ln(1-\hat{p}_0)} = \quad qi \quad \frac{\ln\{1-e^{-[\hat{a}+\hat{b}\varphi(S_i)]}\}}{\ln\{1-e^{-[\hat{a}+\hat{b}\varphi(S_0)]}\}}$.

(5) Testing of acceleration model.

We test the significance of the acceleration model by the correlation coefficient. Consider a pair of samples $(x_i,y_i) = (\phi(x_i),\ln\hat{\theta}_i)$, $i = 1,2,\cdots,k$, and their correlation coefficients are defined as

$$r = \frac{l_{xy}}{(l_{xx}l_{yy})^{1/2}},$$

where l_{xx}, l_{xy} is described above, and

$$l_{yy} = \sum_{i=1}^{k}(y_i - \bar{y})^2 = \sum_{i=1}^{k} y_i{}^2 - \frac{1}{k}(\sum_{i=1}^{k} y_i)^2.$$

For a given significance level $\alpha(0 < \alpha < 1)$, r_α can be found from the critical value table of correlation coefficient test. If $|r| > r_\alpha$, then x_i is considered to be correlated with y_i in this sample, and the acceleration model can be used.

4. An example

We make the statistical analysis of the data from a specific example using the previous statistical methods. After plug-in products are put into production, 80 pieces of qualified products are selected for constant temperature accelerated life test. The four acceleration temperature levels selected are $70^\circ, 90^\circ, 110^\circ, 130^\circ$. It can be seen from similar products that the failure mechanism of this plug-in is the same between 30° and 150°, and the life distributions are all geometric distributions, but the average life is different, and 40° is the normal operating temperature. Arrhenius model is chosen as the acceleration model, i.e., $\ln\theta_i = a + b/S_i$, the unit of S_i here is the absolute temperature. Now 80 sam-

Table 1 Geometrically distributed product constant temperature accelerated life test data

Test conditions	Failure time
$S_1 = 70^\circ\ (343K), n_1 = 20, r_1 = 8$	12,42,153,153,291,302,316,334,595
$S_2 = 90^\circ\ (363K), n_2 = 20, r_2 = 14$	7,18,62,80,80,149,203,247,247,288,295,302,350 359,377,384
$S_3 = 110^\circ\ (383K), n_3 = 20, r_3 = 13$	1,23,45,45,58,77,90,116,124,131,131,146,158 158,165,180
$S_4 = 130^\circ\ (403K), n_4 = 20, r_4 = 13$	3,4,5,8,18,18,20,21,24,25,27,27,35,49,50

ples are divided into four groups, with 20 in each group, and fix-group censored life tests are carried out in a fixed group at four accelerating temperature levels, respectively. The failure groups specified in advance are 8,14,13 and 13, respectively. The failure time is shown in Table 1.

Further data processing is carried out in the following steps.

(1) Testing of acceleration model.

First, the estimated average life value $\hat{\theta}_i$ and its logarithm $y_i = \ln\hat{\theta}_i$ at each temperature level are calculated according to Equation (1). Then the correlation coefficient $r = 0.98898$ between $x_i = S_i^{-1}$ and $y_i = \ln\hat{\theta}_i$ is calculated, and its degree of freedom $f = k - 2 = 4 - 2 = 2$. Check the critical value table of correlation coefficient test, and at the significance level α

= 0.05, the critical value of correlation coefficient test $r_\alpha = 0.95$. From $r > r_\alpha$, it is believed that there is a linear relationship between x_i and y_i.

(2) Estimation of acceleration model.

By the least square method, the estimated values of coefficients a and b in the acceleration model are calculated: $\hat{a} = -13.1644$, $\hat{b} = 6856.722$. Thus the estimation of the accelerated model is obtained as

$$\ln\hat{\theta} = -13.1644 + 6856.722/S.$$

(3) Estimation of reliability indexes.

When S is 40°(313K), the estimated value of reliability indexes such as average life at normal operating temperature can be calculated from the above acceleration equation

$$\hat{\theta}_0 = e^{-13.1644+6856.722/313} = 6260,\ \hat{p}_0 = 1/\hat{\theta}_0 = 1.6 \times 10^{-4},\ \hat{R}_0(1000) = (1 - \hat{p}_0)^{1000} = 0.852366.$$

(4) Estimation of acceleration coefficient.

$$\hat{\tau}_{S_1 \sim S_0} = \frac{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_1)]}\}}{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_0)]}\}} = 6.785921$$

$$\hat{\tau}_{S_2 \sim S_0} = \frac{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_2)]}\}}{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_0)]}\}} = 20.43693,$$

$$\hat{\tau}_{S_3 \sim S_0} = \frac{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_3)]}\}}{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_0)]}\}} = 54.95282,$$

$$\hat{\tau}_{S_4 \sim S_0} = \frac{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_4)]}\}}{\ln\{1 - e^{-[\hat{a}+\hat{b}\varphi(S_0)]}\}} = 134.4631.$$

5. Conclusions

In this paper, we give the maximum likelihood estimation of the parameter of geometric distribution under fix-group and fix-time censored test, set a linear and regression model using the asymptotic normality of MLE, and make the statistical analysis of geometric distribution under constant-stress accelerated life test. Finally we make the statistical analysis of the data from a specific example.

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