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A GENERALIZED LOGISTIC DISTRIBUTION

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Abstract:-
Because of their flexibility, recently, much attention has been given to the study of generalized distributions. A complete study of the transmuted Kumaraswamy Logistic distribution is proposed, introducing some basic properties of this distribution, such as quantile function, characteristic function and entropy are derived, as well as the derivation of maximum likelihood estimates of the parameters and the information matrix, Real life data is used as an application to this distribution with a comparison with other distributions to illustrate the flexibility and ability to model lifetime data. Also, a simulation study is conducted to demonstrate the effect of the sample on the estimates of the parameters.

Keywords:-*Kumaraswamy Logistic distribution, Transmuted distribution, quantile function, entropy and maximum likelihood estimation.*

1. INTRODUCTION

Let G be any valid cumulative distribution function defined on the real line. Many approaches for generating new distributions based on G can be put in the form

$$F(x) = H(G(x)), \qquad (1)$$

where H :[0,1]®[0,1] and F is a valid cumulative distribution function. So, for every G, one can use (1) to generate a new distribution. The first approach of the form (1) in recent years was the exponentiated G distributions due to Mudhokar and Srivastava (1993), Gupta and Kundu (2001), Nassar and Eissa (2003, 2004) and others. The second approach was beta-G distributions due to Eugene et al. (2002), Jones (2004), Nadarajah and Kotz (2004, 2005), Cordeiro and Lemonte (2011), Cordeiro et al. (2012) , Nassar and Nada (2011, 2012 a& b), Nassar and Elmasry (2012) and Mahmoud et al. (20i5), followed by Gamma-G distributions due to Zografos and Balakrishnan (2009). Jones (2009), Cordeiro et al. (2010), Cordeiro and de Castro (2011), Elbatal and Elgarhy (2013), Nassar (2016) and others introduced Kumaraswamy – G distributions important in survival analysis and marketing research.

Transmutation has been receiving increased attention over the last few years, and several transmuted distributions have been investigated such as Shaw and Buckley (2009), Aryal and Tsokos (2009, 2011), Merovci (2013 a& b), Merovci and Puka (2014) and Merovci et al. (2016).

In this paper we introduce a generalization of the Logistic distribution via Kumaraswamy distribution and the Quadratic Rank Transmutation Map, called Transmuted Kumaraswamy Logistic Distribution (TKL). This generalization is flexible enough to model different types of lifetime data important in many areas of research.

Proposing a new model, the so-called Transmuted Kumaraswamy Logistic (TKL) distribution, the article is outlined as follows. In Section 2 we introduce the TKL distribution and provide plots of density function and cumulative distribution function, along with the hazard and survival function. Section 3 introduces some properties of the TKL distribution as well as a complete discussion in deducing an explicit form for the Quantile function and characteristic function followed by the deduction of Renyi and Shannon entropies. In Section 4 we derive the maximum likelihood estimators of the unknown parameters and the Fisher information matrix . We present, in Section 5, a simulation study , followed by an application to real data to illustrate the importance of the TKL distribution, in Section 6.

2. The Transmuted Kumaraswamy Logistic (TKL) Distribution

The Kumaraswamy's distribution on the interval (0 , 1) defined by Kumaraswamy (1980) having the probability density function (pdf) and the cdf with two shape parameters $a >$ 0 and $b > 0$ defined by

$$f(x) = abx^{a-1}(1-x)^{b-1} \qquad and \qquad F(x) = 1-(1-x^a)^b.$$

Jones (2009) construct a new class of KW generalized (KW-G) distributions. From an arbitrary parent cdf $G(x)$, the cdf $F(x)$ of the Kumaraswamy-G distribution is defined by

$$F(x) = 1-[1-G(x)^a]^b, \qquad (2)$$

where $a > 0$ and $b > 0$. Correspondingly, the pdf of this family of distributions has the simple form

$$f(x) = abg(x)G(x)^{a-1}(1-G(x)^a\}^{b-1}, \qquad (3)$$

From (2) the cdf of Kumaraswamy Logistic Distribution is

$$F_{KL}(x) = 1-[1-\left(\frac{1}{1+e^{-\lambda X}}\right)^a]^b, \qquad a,b>0, \lambda>o. \qquad (4)$$

A random variable X is said to have a transmuted distribution *T(x)* if its cumulative distribution function (cdf) defined by Shaw and Buckley (2007) is given by

$$T(x) = (1+t)F(x)\big(1-tF(x)\big), \qquad |t| \le 1. \qquad (5)$$

From (4) and (5) the cdf and pdf of Transmuted Kumaraswamy Logistic Distribution are defined as follows

$$F_{TKL}(x) = \left[1-\left[1-\left(\frac{1}{1+e^{-\lambda X}}\right)^a\right]^b\right]\left[1+t\left[1-\left(\frac{1}{1+e^{-\lambda X}}\right)^a\right]^b\right] \qquad (6)$$

$$a,b>0, \lambda>o \; and \;\; |t| \le 1.$$

$$f_{TKL}(x) = \left[\frac{ab\lambda e^{-\lambda X}}{(1+e^{-\lambda X})^{a+1}}\left[1-\left(\frac{1}{1+e^{-\lambda X}}\right)^a\right]^{b-1}\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X}}\right)^a\right]^b\right]\right] \qquad (7)$$

$$a,b>0, \lambda>o \; and \;\; |t| \le 1.$$

Plots of the pdf and cdf of the TKL for different values of the parameters are given in Figures (1) and (2).

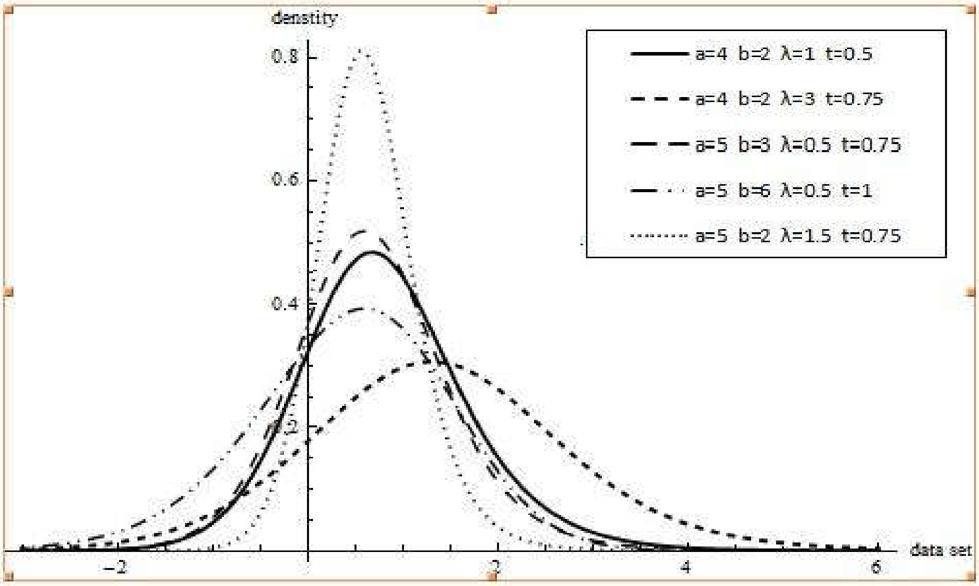

Figure (1). The pdf of the TKL distribution for different values of the parameters.

The survival (reliability) function of the TKL, is defined as

$$S(x) = 1 - F(x) = \left[1 - \left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b \left[1 - t + t\left[1 - \left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b\right], \quad (8)$$

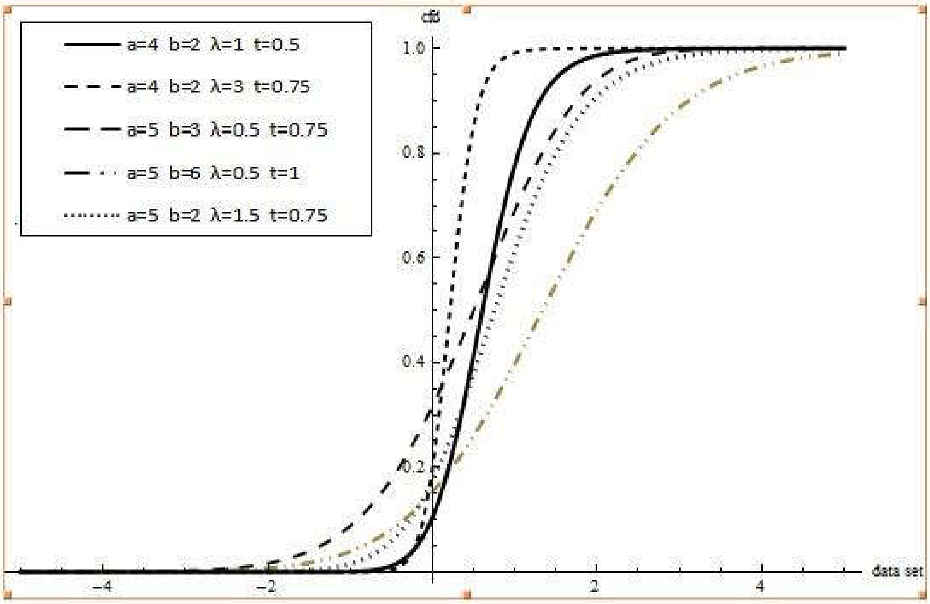

Figure (2). The cdf of the TKL distribution for different values of the parameters.

Now, the hazard rate function of the TKL is given by

$$h(x) = \frac{f(x)}{S(x)}$$

$$= \frac{ab\lambda\, e^{-\lambda x}\left[1 - t + 2t\left[1 - \left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b\right]}{(1+e^{-\lambda x})^{a+1}\left[1 - \left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]\left[1 - t + t\left[1 - \left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b\right]}. \quad (9)$$

Figure (3) illustrates the shape of the hazard rate function for different values of the parameters.

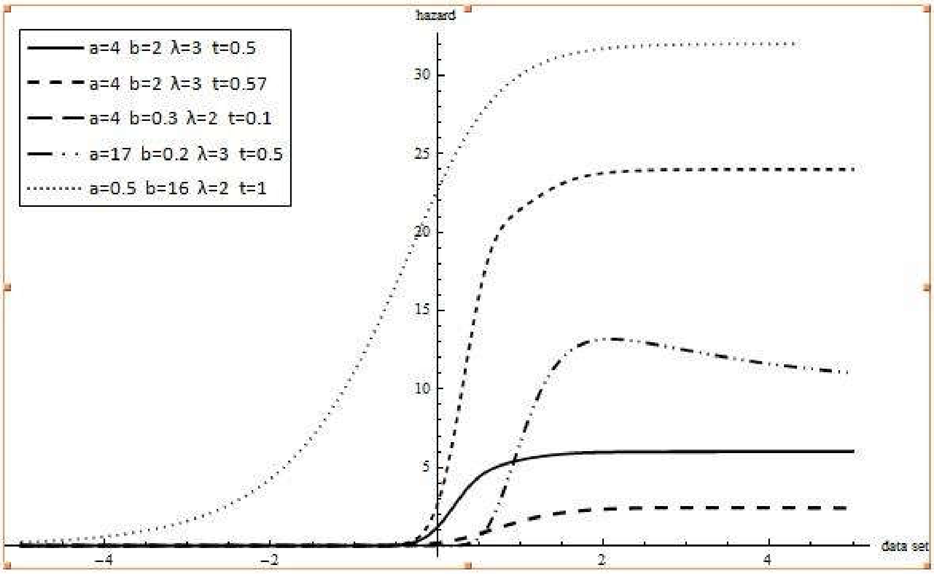

Figure (3). The hazard rate of the TKL distribution for selected values of the parameters.

Section 2 introduces some properties of the $TK(a, b, \lambda, t)$ as studied in the literature as well as a complete discussion in deducing an explicit form for the Quantile function and characteristic function followed by the deduction of Renyi and Shannon entropies. Section 3 we discuss maximum likelihood estimation and determined observed information matrix and expected fisher information matrix. Section 4 we present a simulation study. We conclude, in Section 5, and an application to real data.

3. Properties of the TKL distribution

3.1 Quantile Function

Theorem 1:

Let X be a random variable following $KL(a, b, \lambda, t)$ distribution and let uÎ (0,1) . A value of x such that $F(x)=u$ is called a quantile of order u for the distribution. A quantile of order u is at the following approximate value

$$x \approx \frac{1}{\lambda}\log[\frac{1}{(\frac{b(1+t)}{u})^{1/a}-1}] . \qquad (10)$$

Proof:

Since (x) is continuous and strictly increasing, then the quantile function $x = F^{-1}(u)$, $u \in (0,1)$ can be straightforward computed by inverting (6) to obtain

$$u = \{1-\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b\}\left[1+t\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b\right]$$

$$u = 1-\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b + t\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b - t\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^{2b}$$

$$1-u = (1-t)\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^b + t\left[1-\left(\frac{1}{1+e^{-\lambda x}}\right)^a\right]^{2b}$$

$$1-u = (1-t)\sum_{k=0}^{\infty}\binom{b}{k}(-1)^k\left(\frac{1}{1+e^{-\lambda x}}\right)^{ak} + t\sum_{k=0}^{\infty}\binom{2b}{k}(-1)^k\left(\frac{1}{1+e^{-\lambda x}}\right)^{ak}$$

The summation on the right-hand side converges absolutely for $\left|\frac{1}{1+e^{-\lambda x}}\right| < 1$. Using the approximation technique, then the second approximation, we have

$$1-u \cong (1-t)\left[1-\frac{b}{(1+e^{-\lambda x})^a}\right] + t\left[1-\frac{2b}{(1+e^{-\lambda x})^a}\right]$$

$$1-u \cong 1-\frac{b}{(1+e^{-\lambda x})^a} - t + \frac{tb}{(1+e^{-\lambda x})^a} + t - \frac{2tb}{(1+e^{-\lambda x})^a}$$

$$u = \frac{b}{(1+e^{-\lambda x})^a} + \frac{tb}{(1+e^{-\lambda x})^a}$$

$$= \frac{b}{(1+e^{-\lambda x})^a}(1-t)$$

Therefore, an approximate quantile function of order u of the TKL distribution is given by (10). In particular, the median of the TKL distribution is given by

$$Median \approx \frac{1}{\lambda}\log\frac{1}{\left(2b(1+t)\right)^{1/a}-1} \qquad (11)$$

The random sample can also be easily generated from (7) by taking U as a uniform random variable in (0, 1).

3.2 Characteristic Function

In this subsection, we derive the characteristic function of TKL distribution.

The characteristic function (cf) of the TKL (a, b, λ, t) distribution can be deduced to yield $\Phi(\Theta)_{TKL} = ab(1-t)E(1) + 2$ t

$$\Phi(\Theta)_{TKL} = ab(1-t)E(1) + 2\,\mathrm{t\,ab\,E(2)}, \text{ where} \qquad (12)$$

$$E(i) = \sum_{k=0}^{\infty}(-1)^k\binom{ib-1}{k}B\left(ak+a+\frac{i\theta}{\lambda}, 1-\frac{i\theta}{\lambda}\right), i = 1,2,$$

$$a, b > 0, \lambda > o \text{ and } 0 < t \leq 1 \qquad (13)$$

From equation (12) we can deduce the first and second moments as follows

$$E(X) = \frac{(1-t)}{\lambda}H(1) + \frac{t}{\lambda}H(2),$$

where $H(i) = \sum_{k=0}^{\infty}(-1)^k\binom{2b}{k+1}[\Psi(ak+a)+\gamma]. \; i = 1,2$ (14)

and $E(X^2) = \frac{1-t}{\lambda^2}\,\mathrm{K}(1) + \frac{t}{\lambda^2}K(2)$, where

$$K(i) = \sum_{k=0}^{\infty}(-1)^k\binom{ib}{k+1}\left[\Gamma''(1) - 2\psi(\mathrm{ak+a})\Gamma'(1) + \Gamma(1)\frac{\Gamma''(\mathrm{ak+a})}{\Gamma(\mathrm{ak+a})}\right], i = 1,2$$

where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is known as digamma function. In fact, $\gamma = -\psi(1) = 0.577215$ is called the Euler's constant.

3.3 Renyi and Shannon entropies

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. Since the entropy of a random variable is a measure of variation of the uncertainty, the Renyi entropy can be deduced to yield

$$I_X(\xi) = \frac{1}{1-\xi}\log\{(a\lambda)^{\xi-1}(1-t)^{\xi}b^{\xi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^k\left(\frac{2t}{1-t}\right)^m\binom{\xi-1}{k}\binom{\xi}{m}B\left(bm + \xi(b-1)+1, \frac{k}{a}+\xi\right), \qquad \xi \geq 0, \quad \xi \neq 1 \;. \quad (15)$$

A special case, defined in Shannon's [1948] pioneering work on the mathematical theory of communication, given by Shannon entropy - a major tool in information theory and in almost every branch of science and engineering is

$$\begin{aligned} h_{sh}(f_{TKL}) = &-\log a - \log b - \log\lambda + (1-t)\sum_{k=0}^{\infty}(-1)^k\binom{b}{k+1}[\Psi(ak+a)+\gamma] \\ &+ t\sum_{k=0}^{\infty}(-1)^k\binom{2b}{k+1}[\Psi(ak+a)+\gamma] \\ &- \frac{b(a+1)}{a}\sum_{l=1}^{\infty}\frac{(-1)^l}{l}\left[\frac{(1-t)}{(b+l)} + \frac{2t}{2b+l}\right] \\ &+ (1-b)\{(1-t)[\Psi(b)-\Psi(b+1)] + t\,[\Psi(2b)-\Psi(2b+1)]\} \\ &- \left\{\log(1-t) + \sum_{l=1}^{\infty}\frac{(-1)^l}{l}\frac{(2t)^l}{(1-t)^{l-1}}\left[\frac{1}{(l+1)} + \frac{2t}{(1-t)(l+2)}\right]\right\} \qquad (16) \end{aligned}$$

4. Maximum likelihood estimation

Here, we consider the maximum likelihood estimators (MLE) of the TKL (a, b, λ, t) distribution given in (7). Let X□$(X_1, X_2, ..., X_n)$ be a random sample of size n from this distribution. The log-likelihood function can be written as follows

$$\log L = n\log a + n\log b + n\log\lambda - \lambda\sum_{i=1}^{n} x_i - (ab+1)\sum_{i=1}^{n}\log(1+e^{-\lambda X_i})$$

$$+(b-1)\sum_{i=1}^{n}\log[(1+e^{-\lambda X_i})^{a}-1] + \sum_{i=1}^{n}\log\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\right].$$

Differentiating with respect to a, b, λ and t we obtain the following equations

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} - b\sum_{i=1}^{n}\log(1+e^{-\lambda X_i}) + (b-1)\sum_{i=1}^{n}\frac{\log(1+e^{-\lambda X_i})}{\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]}$$
$$+\sum_{i=1}^{n}\frac{2tb\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b-1}\log(1+e^{-\lambda X})}{(1+e^{-\lambda X_i})^{a}\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\right]},$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} - a\sum_{i=1}^{n}\log(1+e^{-\lambda X_i}) + \sum_{i=1}^{n}\log[(1+e^{-\lambda X_i})^{a}-1]$$
$$+\sum_{i=1}^{n}\frac{2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\log[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}]}{\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\right]},$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i + (ab+1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1+e^{-\lambda X_i}} - (b-1)\sum_{i=1}^{n}\frac{a x_i e^{-\lambda x_i}(1+e^{-\lambda X_i})^{a-1}}{(1+e^{-\lambda X_i})^{a}-1}$$
$$-\sum_{i=1}^{n}\frac{2tbax_i e^{-\lambda x_i}(1-(1+e^{-\lambda X_i})^{-a})^{b-1}}{(1+e^{-\lambda X_i})^{a+1}\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\right]},$$

$$\frac{\partial \log L}{\partial t} = \sum_{i=1}^{n}\frac{2\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}-1}{\left[1-t+2t\left[1-\left(\frac{1}{1+e^{-\lambda X_i}}\right)^{a}\right]^{b}\right]},$$

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The Fisher information matrix$K = (\theta)$, $\theta = (a, b, \lambda, t)^T$, is

$$K = \begin{pmatrix} K_{a,a} & K_{a,b} & K_{a,\lambda} & K_{a,t} \\ K_{b,a} & K_{b,b} & K_{b,\lambda} & K_{b,t} \\ K_{\lambda,a} & K_{\lambda,b} & K_{\lambda,\lambda} & K_{\lambda,t} \\ K_{t,a} & K_{t,b} & K_{t,\lambda} & K_{t,t} \end{pmatrix}$$

whose elements are

$$K_{t,t} = E\left(-\frac{\partial^2 \log L}{\partial t^2}\right) = \frac{n}{1-t}\sum_{j=1}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j\left[\frac{4}{3+j} - \frac{4}{2+j} + \frac{1}{1+j}\right],$$

$$K_{t,a} = E\left(-\frac{\partial^2 \log L}{\partial t\,\partial a}\right) = \frac{-2nb^2}{a(1-t)}\sum_{j=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^j}{l}\left(\frac{2t}{1-t}\right)^j \frac{1}{(b(2+j)+l)(b(2+j)+l-1)},$$

$$K_{t,b} = E\left(-\frac{\partial^2 \log L}{\partial t\,\partial b}\right) = \frac{2nb}{1-t}\sum_{j=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^j}{l}\left(\frac{2t}{1-t}\right)^j B\left(b(2+j),\frac{l}{a}+1\right),$$

$$\begin{aligned} K_{b,b} = E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) &= \frac{n}{b^2} \\ &+ \frac{4nbt^2}{(1-t)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j \Big(B(b(j \\ &+ 3),1)\Big[\big(\Psi(b(j+3)) - \Psi(b(j+3)+1)\big)^2 + \Psi'(b(j+3)) \\ &- \Psi'(b(j+3)+1)\Big]\Big) \\ &- tn\Big[\big(\Psi(2b) - \Psi(2b+1)\big)^2 + \Psi'(2b) - \Psi'(2b+1)\Big], \end{aligned}$$

$$\begin{aligned} K_{b,a} = E\left(-\frac{\partial^2 \log L}{\partial b\,\partial a}\right) &= \frac{nb(1-t)}{a}\left(\frac{1}{b}[\gamma+\Psi(b+1)] + \frac{1}{b-1}[\gamma+\Psi(b)]\right) \\ &- \frac{2t}{1-t}\left(\frac{1}{b}[\gamma+\Psi(2b+1)] + \frac{1}{b-1}[\gamma+\Psi(2b)]\right) \\ &+ \frac{2tbn}{a}B(2b-1,2)[\Psi(2)-\Psi(2b+1)] \\ &+ \frac{4bnt^2}{a(1-t)}\sum_{j=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^j}{l}\left(\frac{2t}{1-t}\right)^j B(b(j+3)+l \\ &- 1,2)[\Psi(b(j+3)+l-1) - \Psi(b(j+3)+l+1)] \end{aligned}$$

$$-\frac{2tnb^2}{a}\sum_{l=1}^{\infty}\frac{1}{l}B(2b+l-1,2)[\Psi(2b+l-1)-\Psi(2b+l+1)],$$

$$\begin{aligned} K_{\lambda,t} = E\left(-\frac{\partial^2 \log L}{\partial\lambda\partial t}\right) &= \frac{2ntb^2a}{\lambda(1-t)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j\left[\sum_{l=1}^{\infty}\frac{-1}{l}\left(2B\left(b(3+j)-1,\frac{l}{a}+2\right) - B\left(2b\right.\right.\right. \\ &\left.+j-1,\frac{l}{a}+2\right) - 2B\left(b(3+j)-1,\frac{l+1}{a}+2\right) \\ &\left.+ B\left(b(2+j)-1,\frac{l+1}{a}+2\right)\right) \\ &+\frac{1}{a}\Big(2B(b(3+j)-1,2)[\Psi(2)-\Psi(b(3+j)+1)] \\ &- B(b(2+j)-1,2)[\Psi(2)-\Psi(b(2+j)+1)] \\ &- 2B\left(b(3+j)-1,\frac{1}{a}+2\right)\left[\Psi\left(\frac{1}{a}+2\right) - \Psi\left(b(3+j)+\frac{1}{a}+1\right)\right] \\ &\left.+ B\left(b(2+j)-1,\frac{1}{a}+2\right)\left[\Psi\left(\frac{1}{a}+2\right) - \Psi\left(b(2+j)+\frac{1}{a}+1\right)\right]\Big)\right) \\ &+ \frac{2nb^2}{\lambda}\left(\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(2b-1,\frac{l}{a}\right) - B\left(2b-1,\frac{l+1}{a}+2\right)\right]\right. \\ &+\frac{1}{a}\Big[B(2b-1,2)[\Psi(2)-\Psi(2b+1)] + \left(2b-1,\frac{1}{a}+2\right) \\ &\left.-\Psi\left(2b+\frac{1}{a}+1\right)\Big]\right]. \end{aligned}$$

$$\begin{aligned} K_{aa} = E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) &= \frac{n}{a^2} \\ &+ \frac{b(b-1)n}{a^2}\Big((1-t)B(b-2,2)\Big[\big(\Psi(b-2)-\Psi(b)\big)^2 + \Psi'(b-2) \\ &- \Psi'(b)\Big] \\ &+ 2tB(2b-2,2)\Big[\big(\Psi(2b-2)-\Psi(2b)\big)^2 + \Psi'(2b-2) - \Psi'(2b)\Big]\Big) \end{aligned}$$

$$+\frac{2ntb^2}{a^2}\Bigg(\frac{2bt}{(1-t)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j B(b(j+3)$$
$$-2,3)\left[\left(\Psi(b(j+3)-2)-\Psi(b(j+3)+1)\right)^2+\Psi'(b(j+3)-2)\right.$$
$$\left.-\Psi'(b(j+3)+1)\right]$$
$$-(b-1)B(2b-2,3)\left[\left(\Psi(2b-2)-\Psi(2b+1)\right)^2+\Psi'(2b-2)\right.$$
$$\left.-\Psi'(2b+1)\right]$$
$$+B(2b-1,2)\left[\left(\Psi(2b-1)-\Psi(2b+1)\right)^2+\Psi'(2b-1)-\Psi'(2b+1)\right]\Bigg).$$

$$K_{b,\lambda}=E\left(-\frac{\partial^2\log L}{\partial b\partial\lambda}\right)=$$
$$-\frac{anb(1-t)}{\lambda}\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(b,\frac{l}{a}+1\right)+\frac{2t}{1-t}B\left(2b,\frac{l}{a}+1\right)-B\left(b,\frac{l+1}{a}+1\right)\right.$$
$$\left.+\frac{2t}{1-t}B\left(2b,\frac{l+1}{a}+1\right)\right]$$
$$+\frac{n(1-t)}{\lambda}\left[[\gamma+\Psi(b+1)]+\frac{t}{1-t}[\gamma+\Psi(2b+1)]\right.$$
$$+b\,B\left(b,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(b+\frac{1}{a}+1\right)\right]$$
$$\left.-\frac{2tb}{1-t}B\left(2b,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(2b+\frac{1}{a}+1\right)\right]\right]$$
$$+\frac{nab(1-t)}{\lambda}\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(b-1,\frac{l}{a}+1\right)+\frac{2t}{1-t}B\left(2b-1,\frac{l}{a}+1\right)-B\left(b-1,\frac{l}{a}+\frac{1}{a}+1\right)\right.$$
$$\left.+\frac{2t}{1-t}B\left(2b-1,\frac{l}{a}+\frac{1}{a}+1\right)\right]$$
$$-\frac{nb(1-t)}{\lambda}\left[\frac{1}{(b-1)}[\gamma+\Psi(b)]+\frac{2t}{(1-t)(2b-1)}[\gamma+\Psi(2b)]\right.$$
$$+B\left(b-1,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(b+\frac{1}{a}\right)\right]$$
$$\left.+\frac{2t}{1-t}B\left(2b-1,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(2b+\frac{1}{a}\right)\right]\right]$$
$$+\frac{2tanb}{\lambda}\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-1,\frac{l}{a}+2\right)+\frac{1}{a}B(2b-1,2)[\Psi(2)-\Psi(2b+1)]\right.$$
$$-\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-1,\frac{l+1}{a}+2\right)$$
$$\left.-\frac{1}{a}B\left(2b-1,\frac{1}{a}+2\right)\left[\Psi\left(\frac{1}{a}+2\right)-\Psi\left(2b+\frac{1}{a}+1\right)\right]\right]$$
$$-\frac{2tanb^2}{\lambda(1-t)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j\Big[\sum_{l=1}^{\infty}\frac{1}{l}\Big[B\left(bj+3b-1,\frac{l}{a}+2\right)\Big[\Psi(b(j+3)$$
$$-1)-\Psi\left(b(j+3)+\frac{l}{a}+1\right)\Big]$$
$$-B\left(b(j+3)-1,\frac{l+1}{a}+2\right)\Big[\Psi(b(j+3)-1)$$
$$-\Psi\left(b(j+3)+\frac{l+1}{a}+1\right)\Big]\Big]$$
$$-\frac{1}{a}\sum_{m=1}^{\infty}\frac{1}{m}\Big[B(b(j+3)+m-1,2)[\Psi(b(j+3)+m-1)$$
$$-\Psi(b(j+3)+m+1)]$$
$$-B\left(b(j+3)+m-1,\frac{1}{a}+2\right)\Big[\Psi(b(j+3)+m-1)$$
$$-\Psi\left(b(j+3)+m+\frac{1}{a}+1\right)\Big]\Big]$$
$$+\frac{anb^2}{\lambda}\left[\sum_{l=1}^{\infty}\frac{1}{l}\left(B\left(2b-1,\frac{l}{a}+2\right)\left[\Psi(2b-1)-\Psi\left(2b+\frac{l}{a}+1\right)\right]\right.\right.$$
$$\left.-B\left(2b-1,\frac{l+1}{a}+2\right)\left[\Psi(2b-1)-\Psi\left(2b+\frac{l+1}{a}+1\right)\right]\right)$$
$$-\frac{1}{a}\sum_{m=1}^{\infty}\frac{1}{m}\Big(B(2b+m-1,2)[\Psi(2b+m-1)-\Psi(2b+m+1)]$$
$$\left.-B\left(2b+m-1,+\frac{1}{a}+2\right)\left[\Psi(2b+m-1)-\Psi\left(2b+m+\frac{1}{a}+1\right)\right]\Big)\right].$$

$$K_{a,\lambda} = E\left(-\frac{\partial^2 \log L}{\partial a \partial \lambda}\right)$$

$$= \frac{-nb^2}{\lambda}\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(b,\frac{l}{a}+1\right)+\frac{2t}{1-t}B\left(2b,\frac{l}{a}+1\right)-B\left(b,\frac{l+1}{a}+1\right)\right.$$

$$\left.+\frac{2t}{1-t}B\left(2b,\frac{l+1}{a}+1\right)\right]$$

$$+\frac{n\,b(1-t)}{\lambda a}\left[[\gamma+\Psi(b+1)]+\frac{t}{1-t}[\gamma+\Psi(2b+1)]\right.$$

$$+b\,B\left(b,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(b+\frac{1}{a}+1\right)\right]$$

$$\left.-\frac{2tb}{1-t}B\left(2b,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(2b+\frac{1}{a}+1\right)\right]\right]$$

$$+\frac{nb(b-1)(1-t)}{\lambda}\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(b-1,\frac{l}{a}+1\right)+\frac{2t}{1-t}B\left(2b-1,\frac{l}{a}+1\right)\right.$$

$$\left.-B\left(b-1,\frac{l+1}{a}+1\right)+\frac{2t}{1-t}B\left(2b-1,\frac{l+1}{a}+1\right)\right]$$

$$-\frac{nb(b-1)(1-t)}{\lambda a}\left[\frac{1}{(b-1)}[\gamma+\Psi(b)]+\frac{2t}{(1-t)(2b-1)}[\gamma+\Psi(2b)]\right.$$

$$+B\left(b-1,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(b+\frac{1}{a}\right)\right]$$

$$\left.+\frac{2t}{1-t}B\left(2b-1,\frac{1}{a}+1\right)\left[\Psi\left(\frac{1}{a}+1\right)-\Psi\left(2b+\frac{1}{a}\right)\right]\right]$$

$$-\frac{nb(b-1)(1-t)}{\lambda}\left[\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(b-2,\frac{l}{a}+2\right)\left[\Psi\left(\frac{l}{a}+2\right)-\Psi\left(b+\frac{l}{a}\right)\right]\right.\right.$$

$$\left.+\frac{1}{a}B(b-2,2)\left[\left(\Psi(2)-\Psi(b)\right)^2+\Psi'(2)-\Psi'(b)\right]\right]$$

$$+\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(b-2,\frac{l+1}{a}+2\right)\left[\Psi\left(\frac{l+1}{a}+2\right)-\Psi\left(b+\frac{l+1}{a}\right)\right]\right.$$

$$+\frac{1}{a}B\left(b-2,\frac{1}{a}+2\right)\left[\left(\Psi\left(\frac{1}{a}+2\right)-\Psi\left(b+\frac{1}{a}\right)\right)^2+\Psi'\left(\frac{1}{a}+2\right)\right.$$

$$\left.\left.\left.-\Psi'\left(b+\frac{1}{a}\right)\right]\right]\right]$$

$$-\frac{2t}{1-t}\left[\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-2,\frac{l}{a}+2\right)\left[\Psi\left(\frac{l}{a}+2\right)-\Psi\left(2b+\frac{l}{a}\right)\right]\right.\right.$$

$$\left.+\frac{1}{a}B(2b-2,2)\left[\left(\Psi(2)-\Psi(2b)\right)^2+\Psi'(2)-\Psi'(2b)\right]\right]$$

$$-\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-2,\frac{l+1}{a}+2\right)\left[\Psi\left(\frac{l+1}{a}+2\right)-\Psi\left(2b+\frac{l+1}{a}\right)\right]\right.$$

$$+\frac{1}{a}B\left(2b-2,\frac{1}{a}+2\right)\left[\left(\Psi\left(\frac{1}{a}+2\right)-\Psi\left(2b+\frac{1}{a}\right)\right)^2+\Psi'\left(\frac{1}{a}+2\right)\right.$$

$$\left.\left.\left.-\Psi'\left(2b+\frac{1}{a}\right)\right]\right]\right]$$

$$-\frac{2tnb^2}{\lambda}\left[\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-1,\frac{l}{a}+2\right)+B\left(2b-1,\frac{l+1}{a}+2\right)\right.$$

$$+\frac{1}{a}\left[-B(2b-1,2)[\Psi(2)-\Psi(2b+1)]\right.$$

$$\left.\left.+B\left(2b-1,\frac{1}{a}+2\right)\left[\Psi\left(\frac{1}{a}+2\right)-\Psi\left(2b+\frac{1}{a}+1\right)\right]\right]\right]$$

$$-\frac{4b^3t^3n}{\lambda(1-t)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2t}{1-t}\right)^j\Big[\sum_{l=1}^{\infty}\frac{-1}{l}\left[B\left(b(j+3)-2,\frac{l}{a}+3\right)\left[\Psi\left(\frac{l}{a}+3\right)\right.\right.$$

$$\left.-\Psi\left(b(j+3)+\frac{l}{a}+1\right)\right]$$

$$-\frac{1}{a}B(b(j+3)-2,3)\left[\left(\Psi(3)-\Psi(b(j+3)+1)\right)^2+\Psi'(3)\right.$$

$$\left.\left.-\Psi'(b(j+3)+1)\right]\right]$$

$$+\sum_{l=1}^{\infty}\frac{1}{l}\left[B\left(bj+3b-2,\frac{l+1}{a}+3\right)\left[\Psi\left(\frac{l+1}{a}+3\right)-\Psi\left(b(j+3)+\frac{l+1}{a}+1\right)\right]\right.$$

$$+\frac{1}{a}B\left(b(j+3)-2,3+\frac{1}{a}\right)\left[(\Psi\left(3+\frac{1}{a}\right)-\Psi\left(b(j+3)+\frac{1}{a}+1\right))^2\right.$$

$$\left.\left.+\Psi'\left(3+\frac{1}{a}\right)-\Psi'\left(b(j+3)+\frac{1}{a}+1\right)\right]\right]$$

$$+\frac{2nb^2a(a-1)}{\lambda}\left[\sum_{l=1}^{\infty}\frac{-1}{l}B\left(3b-2,\frac{l}{a}+3\right)\left[\Psi\left(\frac{1}{a}+3\right)-\Psi\left(3b+\frac{1}{a}+1\right)\right]\right.$$

$$+\frac{1}{a}B(3,3b-2)[(\Psi(3)-\Psi(3b+1))^2+\Psi'(3)-\Psi'(3b+1)]$$

$$+\sum_{l=1}^{\infty}\frac{1}{l}B\left(3b-2,\frac{l+1}{a}+3\right)\left[\Psi\left(\frac{l+1}{a}+3\right)-\Psi\left(3b+\frac{l+1}{a}+1\right)\right]$$

$$+\frac{1}{a}B\left(\frac{1}{a}+3,3b-2\right)\left[(\Psi\left(\frac{1}{a}+3\right)-\Psi\left(3b+\frac{1}{a}+1\right))^2+\Psi'\left(\frac{1}{a}+3\right)\right.$$

$$\left.\left.-\Psi'\left(3b+\frac{1}{a}+1\right)\right]\right]$$

$$-\frac{2tnb^2}{\lambda}\left[\sum_{l=1}^{\infty}\frac{-1}{l}B\left(2b-1,\frac{l}{a}+2\right)\left[\Psi\left(\frac{1}{a}+2\right)-\Psi\left(2b+\frac{1}{a}+1\right)\right]\right.$$

$$-\frac{1}{a}B(2,2b-1)[(\Psi(2)-\Psi(2b+1))^2+\Psi'(2)-\Psi'(2b+1)]$$

$$+\sum_{l=1}^{\infty}\frac{1}{l}B\left(2b-1,\frac{l+1}{a}+2\right)\left[\Psi\left(\frac{l+1}{a}+2\right)-\Psi\left(2b+\frac{l+1}{a}+1\right)\right]$$

$$+\frac{1}{a}B\left(\frac{1}{a}+2,2b-1\right)\left[(\Psi\left(\frac{1}{a}+2\right)-\Psi\left(2b+\frac{1}{a}+1\right))^2+\Psi'\left(\frac{1}{a}+2\right)\right.$$

$$\left.\left.-\Psi'\left(2b+\frac{1}{a}+1\right)\right]\right],$$

$$K_{\lambda,\lambda}=E\left(-\frac{\partial^2\log L}{\partial\lambda^2}\right)=\frac{n}{b^2}$$

$$-\frac{4nb(ab+1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left[(1-t)\left(B\left(b,\frac{l}{a}+1\right)\right.\right.$$

$$\left.-2B\left(b,\frac{l+1}{a}+1\right)+B\left(b,\frac{l+2}{a}+1\right)\right)$$

$$\left.+2t\left(B\left(2b,\frac{l}{a}+1\right)-2B\left(2b,\frac{l+1}{a}+1\right)+B\left(2b,\frac{l+2}{a}+1\right)\right)\right]$$

$$+\frac{nb(ab+1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left[(1-t)\left(B\left(b,\frac{l}{a}+1\right)\right.\right.$$

$$\left.\left.-B\left(b,\frac{l+1}{a}+1\right)\right)+2t\left(B\left(2b,\frac{l}{a}+1\right)-B\left(2b,\frac{l+1}{a}+1\right)\right)\right]$$

$$+\frac{nb(b-1)a^2}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left[(1-t)\left(B\left(b-2,\frac{l}{a}+1\right)\right.\right.$$

$$\left.-2B\left(b-2,\frac{l+1}{a}+1\right)+B\left(b-2,\frac{l+2}{a}+1\right)\right)$$

$$+2t\left(B\left(2b-2,\frac{l}{a}+1\right)-2B\left(2b-2,\frac{l+1}{a}+1\right)\right.$$

$$\left.\left.+B\left(2b-2,\frac{l+2}{a}+1\right)\right)\right]$$

$$-\frac{abn(a-1)(b-1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left[(1\right.$$

$$\left.-t)\left(B\left(b-1,\frac{l}{a}+1\right)-2B\left(b-1,\frac{l+1}{a}+1\right)+B\left(b-1,\frac{l+2}{a}+1\right)\right)\right.$$

$$+2t\left(B\left(2b-1,\frac{l}{a}+1\right)-2B\left(2b-1,\frac{l+1}{a}+1\right)\right.$$

$$\left.\left.+B\left(2b-1,\frac{l+2}{a}+1\right)\right)\right]$$

$$
-\frac{anb(b-1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\Bigg[(1-t)\bigg(B\left(b-1,\frac{l}{a}+1\right)
- B\left(b-1,\frac{l+1}{a}+1\right)\bigg)+2t\bigg(B\left(2b-1,\frac{l}{a}+1\right)-B\left(2b-1,\frac{l+1}{a}+1\right)\bigg)\Bigg]
$$

$$
+\frac{4a^2b^2t^2n}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-2)^l(-1)^u}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left(\frac{2t}{1-t}\right)^u\Bigg[B\left(3b+bu
-2,\frac{l}{a}+3\right)+B\left(b(3+u)-2,\frac{l+2}{a}+3\right)-2B\left(b(3+u)-2,\frac{l+1}{a}+3\right)\Bigg]
$$

$$
-\frac{2a^2btn(b-1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-2)^l(-1)^u}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\left(\frac{2t}{1-t}\right)^u\Bigg[B\left(b(2+u)
-2,\frac{l}{a}+3\right)+B\left(b(2+u)-2,\frac{l+2}{a}+3\right)-2B\left(b(2+u)-2,\frac{l+1}{a}+3\right)\Bigg]
$$

$$
+\frac{2abtn(a+1)}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\Bigg[B\left(2b-1,\frac{l}{a}+2\right)
-2B\left(2b-1,\frac{l+1}{a}+2\right)+2B\left(2b-1,\frac{l+2}{a}+2\right)\Bigg]
$$

$$
-\frac{2abtn}{\lambda^2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^l}{(2m-1)(2j-1)}\binom{2m+2j-2}{l}\Bigg[B\left(2b-1,\frac{l}{a}+2\right)
-B\left(2b-1,\frac{l+1}{a}+2\right)\Bigg].
$$

The MLE $\hat{\theta} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{t}_{ML})$ of□ is determined from the solution of the nonlinear system of equations given earlier. Under conditions that are fulfilled for the parameter □ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $[\sqrt{n}(\hat{a}_{ML}-a), \sqrt{n}(\hat{b}_{ML}-b), \sqrt{n}(\hat{\lambda}_{ML}-\lambda), \sqrt{n}(t_{ML}-t)]^T$ is $N_4(0, K^{-1}(a, b, \lambda, t)^T)$. The asymptotic normal $_4(0, K^{-1}(\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{t}_{ML})^T)$ distribution of $\hat{\theta} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{t}_{ML})^T$ can be used to construct confidence regions for some parameters and for the hazard and survival functions. In fact, a100(1□□)% asymptotic confidence interval (ACI) for each parameter is given by

$$ACI_a = \left(\hat{a}_{ML} - z_{\gamma/2}\sqrt{K_{11}},\ \hat{a}_{ML} + z_{\gamma/2}\sqrt{K_{11}}\right),$$

$$ACI_b = \left(\hat{b}_{ML} - z_{\gamma/2}\sqrt{K_{22}},\ \hat{b}_{ML} + z_{\gamma/2}\sqrt{K_{22}}\right),$$

$$ACI_\lambda = \left(\hat{\lambda}_{ML} - z_{\gamma/2}\sqrt{K_{33}},\ \hat{\lambda}_{ML} + z_{\gamma/2}\sqrt{K_{33}}\right),$$

$$ACIt = \left(t_{ML} - z_{\gamma/2}\sqrt{K_{44}},\ \hat{t}_{ML} + z_{\gamma/2}\sqrt{K_{44}}\right).$$

where K_{ii} denotes the i[th] diagonal element of $K^{-1} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{t}_{ML})^T$ for I = 1, 2 ,3,4 and $z_{□/2}$ is the 1□□/2 of the standard normal distribution.

5. Simulation Study

We conducted Mont Carlo simulation studies to assess the finite sample behavior of the TKL (, b, λ, t). All results we obtained from 1000 Mont Carlo replication simulations. The TKL random number generation was performed using the inversion method. In each replication, random sample of size n is drawn from the TKL(a, b, λ, t) distribution and the maximum likelihood estimates (MLEs) of the parameters were obtained. The mean, variance, bias and mean squared error (MSE) for each parameter was computed under different sample size n=25, 50, 75, 100, and 200.

Table (1): Mean estimates, bias, variance and mean square errors of the (MLEs) when a=3, b=2, λ=5, t=0.8.

N	parameter	Mean	Variance	Bias	MSE
25	a b λ t	2.8578 0.766405 14.01822 0.749882	1.852897 0.373454 5.957317 0.090219	1.4168 -0.75256 4.82880 -0.14344	3.86 .9398 29.274 0.111
50	a b λ t	4.27097 2.26156 7.36970 0.82383	1.805647005 5.677321255 8.083446715 0.005201342	1.270972857 0.261563429 2.369701429 0.023831143	3.421019008 5.745736682 13.69893158 0.005769266
75	a b λ t	3.61447 1.028334 10.72872 0.540939	0.902256264 0.526750166 15.72706698 0.091674244	0.61447 -0.9716659 5.72872 - 0.2590609	1.279829645 1.470884703 48.54529982 0.158786771
100	A b λ t	4.47547 2.047784 7.211599 0.772704	0.90896 1.8670 7.1216 0.11102	1.47547 0.04778 2.211598 -0.0277296	3.085116157 1.869306586 12.01279725 0.111765887
200	A b λ t	3.747577 1.088347 8.994353 0.721761	0.18687 0.036533 2.831325 0.16657	0.74757 -0.911652 3.94435 -0.078238	0.745750511 0.867643584 18.78618042 0.172700389

We note that the variance and the MSE of the parameters a, b, λ and t decrease as the sample size increases. The mean estimates of the parameters tend to be closer to the true parameter values. It is observed that for all values of n, the variance and MSE of the estimator of t are small as expected.

6. Numerical example

In this section, we study the transmuted Kumaraswamy logistic distributions and provide detailed mathematical treatment for this distribution. As an application, consider short- and long-term outcomes of constraint-induced movement therapy after stroke investigated in a randomized controlled feasibility trial by Dahl et al. (2008). The 30 patients were assessed at baseline, post treatment, and a 6-month follow-up using the Wolf Motor Function Test as primary outcome measure. The test consists of 17 tasks with 2 strength and 15 timed tasks which vary from gross shoulder movements to complex finger grips. The measurement was done by the analysis of videotapes. The 30 observations were 0.5, 1.0, 1.0, 1.5, 1.0, 1.5, 2.0, 1.0, 0.5, 1.0, 0.5, 1.0, 1.0, 1.5, 1.0, 0.5, 1.0, 1.5, 1.0, 1.0, 0.5, 1.0, 1.0, 1.5, 1.5, 1.0, 1.0, 0.5, 1.0, and 1.0, measured in seconds.

The maximum likelihood method is applied to estimate the parameters of the three models Logistic (L), Kumaraswamy Logistic (KL) and Transmuted Kumaraswamy Logistic (TKL) distribution. The resulting estimates with the negative of the likelihood function ($-\ell$).

Table (2). The maximum likelihood estimated and Log- likelihood function for the data set.

Model	maximum likelihood estimates	$-\ell$
L	$\hat{\lambda}$= 1.8222	50.360
KL	$\hat{a}$= 8.40943 $\hat{b}$= 2.47759 $\hat{\lambda}$= 2.04935	41.994
TKL	$\hat{a}$= 12.779 $\hat{b}$= 2.78024 $\hat{\lambda}$= 2.00713 $\hat{t}$= 0.116139	34.538

The variance covariance matrix $(\theta)^{-1}$ of the MLEs under the TKL distribution for the data set is computed as

$$\begin{pmatrix} 0.2555 & 0.9712 & -0.4794 & 0.0323 \\ 0.9712 & 2.744 & -0.2645 & -0.3792 \\ -0.4794 & -0.2645 & 0.0382 & -0.0020 \\ 0.0323 & -0.3793 & -0.0020 & 0.0691 \end{pmatrix}$$

Thus $var\ (\hat{a}) = 0.2555$, $var\ (\hat{b}) = 2.744$, $var(\hat{\lambda}) = 0.0382$, $var(\hat{t}) = 0.0691$. There for, 95% confidence interval for a. b, λ and t are [11.788, 13.769], [-0.467, 6.027], [1.624, 2.390], [-0.504, 0.631], respectively.

Table (3): Criteria comparison for data set

Model	$-\ell$	AIC	AICC	BIC
L	50.360	102.72	102.863	102.197
KL	41.994	89.988	90.911	88.419
TKL	34.538	42.538	44.138	74.984

In order to compare the three distributions, we consider criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC the Bayesian information criterion, for the data set given by Dahl et al. (2008). As shown in table (1.3) the better distribution corresponds to smaller $-\ell$, AIC, AICC and BIC values where

where

$$AIC = 2K - 2\ell,$$

$$\text{AICC} = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k\ log\ n - 2\ell,$$

Here k is the number of parameters and n is the number of observations. The values of the parameters' estimates are used to plot the pdf for the three distributions L, KL and TKL in Fig.(4)

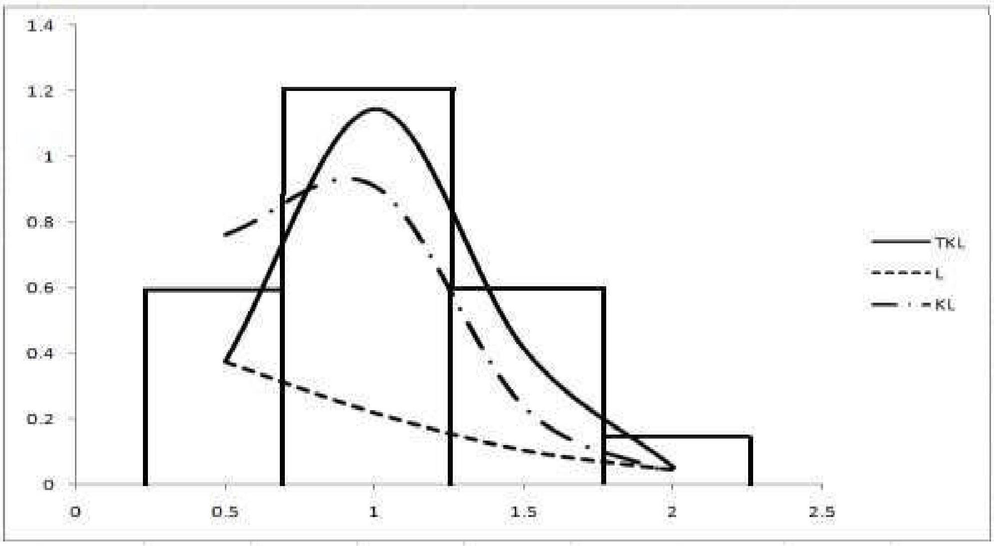

Figure (4). Estimated densities of the models for data set.

7. Concluding Remarks

In this paper, we proposed a new distribution, named the transmuted Kumaraswamy Logistic distribution which extends the Kumaraswamy Logistic distribution. Several properties of the new distribution were investigated, including moments, median, Rényi and Shannon entropy. The model parameters are estimated by maximum likelihood and the information matrix is derived. An application of the transmuted Kumaraswamy Logistic distribution (TKL) to real data is considered. The results of our study indicate that the TKL distribution has the lowest AIC, AICC and BIC statistics among all the sub-models. From the plots of the fitted densities and histogram, Cleary, the TKL distribution provides a closer fit to the histogram than the other Logistic and Kumaraswamy Logistic model. Therefore, the new TKL model can be used quite effectively in analyzing data. Also, we note that the Monte Carlo simulation indicate that the performance of the maximum likelihood estimation are quite satisfactory. Finally, the application to the real data sets shows that the fit of the new model is superior to the fits of its main sub- models. We hope that the proposed model can be used effectively to as a competitive model to fit real data.

References

[1]. G. R. Aryal (2013) "Transmuted log-logistic distribution," *Journal of Statistics Applications & Probability,* vol. 2, no.1, pp. 11-20.

[2]. G. R. Aryal, and C. P. Tsokos, (2009) "On the transmuted extreme value distribution with application," *Nonlinear Analysis: Theory, Methods & Applications*, vol.71, no. 12, pp. 1401-1407, 2009.

[3]. G. R. Aryal and C. P. Tsokos (2011) "Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution." *European Journal of Pure and Applied Mathematics*, 4, 89– 102.

[4]. Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation,* 81 (7), 883-898.
[5]. G. M. Cordeiro and A.J. Lemonte (2011)"The beta Laplace distribution", *Statistics and Probability Letters* 81(8), 973-982.
[6]. G. M. Cordeiro, E. M. M. Ortega and S. Nadarajah (2010) "The Kumaraswamy Weibull distribution with application to failure data" Journal of the Franklin Institute 347 (8), 13991429.
[7]. G. M. Cordeiro, S. Nadarajah, E.M. M. Ortega (2012)" The Kumaraswamy Gumbel distribution" *Statistical Methods & Applications,* 21 (2), 139-168.
[8]. Dahl, A. E., Askim, T., Stock, R., Langergen, E., Lydersen, S. and Indredavik, B. (2008)"Short- and long-term outcome of constraint induced movement therapy after stroke: a randomized controlled feasibility trial", *Clinical Rehabilitation* 22, 436-447.
[9]. I. Elbatal and M. Elgarhy (2013). "Transmuted Quasi Lindley Distributions: A generalization of the Quasi Lindley Distribution", *International Journal of Pure and Applied Sciences and Technology*, 18(2), pp. 59–69.
[10]. N. Eugene, C. Lee and F. Famoye (2002) "Beta-normal distribution and its applications", *Communications in Statistics- Theory and Methods,* **31** (4), 497-512.
[11]. R. D. Gupta and D. Kundu, (2001)."Exponentiated exponential family: an alternative to gamma and Weibull" *Biometrical Journal* 43, 117-130.
[12]. M. C. Jones (2004) "Families of distributions arising from distributions of order statistics" *Tests,* **13** (1), 1-43.
[13]. M. C. Jones (2009) " Kumaraswamy distribution: A beta – type Distribution with some tractability advantages*",* *Statistical Methodology*, 6(1), 70-81.
[14]. P. Kumaraswamy (1980)" Generalized probability density-function for double-bounded random-processes" *Journal of Hydrology*, 462, 79–88.
[15]. M. Mahmoud, M. M.Nassar and M. A. Aefa (2015) "A new generalization of Kumaraswamy Lindley distribution" *Journal of Statistics- Advances in Theory and Applications,* 14, 69-105.
[16]. F. Merovci (2013a) "Transmuted Rayleigh Distribution" *Austrian Journal of Statistics*, 42(1): 2131.
[17]. F. Merovci (2013b) "Transmuted generalized Rayleigh distribution" *Journal of Statistics Applications and Probability*, Volume 2, No. 3, 1-12.
[18]. Merovci, and L. Puka, (2014) "Transmuted Pareto distribution" *ProbStat*, 7, 1-11.
[19]. Merovci, M. Alizadeh and G.G .Hamedani (2016) "Another generalized transmuted family of distributions: properties and applications" *Austrian Journal of Statistics* 45, 71-93.
[20]. S. Mudholkar, and D. K. Srivastava, (1993) "Exponentiated Weibull family for analyzing bathtub failure-rate data" *IEEE Transactions on Reliability* 42, 299-302.
[21]. S. Nadarajah and S. Kotz (2004) "The beta Gumbel distribution" *Mathematics and Probability for Engineers* 10 , 323–332.
[22]. S. Nadarajah and S. Kotz (2005) "The beta exponential distribution", *Reliability Engineering and System Safety* 91 , 689–697.
[23]. M. Nassar and F. H. Eissa (2003) "On the exponentiated Weibull distribution" *Communications in Statistics – Theory and Methods*, Vol. 32, No. 7, 1317-1336.
[24]. M**.** Nassar and F. H. Eissa (2004) "Bayesian estimation for the exponentiatedWeibull model" *Communications in Statistics – Theory and Methods*, Vol. 33, No. 10, 2343-2362..
[25]. M.M. Nassar, N.K. Nada,(2011) The beta generalized Pareto distribution, *Journal of Statistics – Advances in Theory and Applications* 6 , 1–17.
[26]. M. M. Nassar and N. K. Nada (2012) "A new generalization of the exponential geometric distribution" *Journal of Statistics- Advances in Theory and Applications*, 7, 25-48.
[27]. M. M. Nassar and A. Elmasry (2012) "A study of generalized Logistic distributions" *Journal of Egyptian Mathematical Society*, 20(2),126-133 .
[28]. M. M. Nassar and N. K. Nada (2013) "A new generalization of the Pareto geometric distribution" *Journal of Egyptian Mathematical Society*, 21(2),148-155.
[29]. M. M. Nassar (2016) "The Kumaraswamy Laplace Distribution" *Pakistan Journal of Statistics and Operation Research*, XII(4), 609-624.
[30]. C. E. Shannon (1948) "A mathematical theory of communication", *Bell Syst. Tech. J* 27, 379 432.
[31]. W. Shaw and I. Buckley (2007) "The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map," *Research Report*. View at Google Scholar.
[32]. Zografos, K. and Balakrishnan, N. (2009) "On families of beta- and generalized gamma-generated distributions and associated inference", *Statistical Methodology*, 6, 344-362.