



EXISTENCE RESULTS FOR BIGENERAL VECTOR EQUILIBRIUM PROBLEM ON BICONVEX SETS: A KKM APPROACH

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Abstract

In this paper, a general vector equilibrium problem and a general strong vector equilibrium problem are introduced and studied in the Hausdorff topological vector spaces, along with many existing models of vector equilibrium problems and equilibrium problems as special cases. We also consider and introduce some new concepts of general biconvex sets and functions involving an arbitrary function. Applying the KKM theorem yields an existence conclusion for the bi general vector equilibrium problem and the bigeneral strong vector equilibrium problem. There are also a few particular examples and cases displayed.

Keywords

Bigeneral Vector equilibrium problem, Bigeneral P-monotonicity, General biconvexity, KKM-mappings, Existence theory.

1. Introduction

The equilibrium problem, which is a generalization of standard variational inequalities, has been the focus of extensive modern studies. The problem may be traced back to Blum and Oettli [9] and Noor and Oettli [26]. Since vector equilibrium problems provide frameworks for a variety of problem classes, including vector variational inequality, vector complementarity, vector optimization, and vector saddle point problems, many researchers are becoming more interested in them (see [9, 12, 13, 26] and references therein). For vector (quasi-equilibrium problems, a number of eminent scholars have established existence results, including [4, 8, 15, 18, 19, 27]. General biconvex functions are the title given to this specific class of nonconvex functions [24]. We examine a different class of nonconvex functions with regard to an arbitrary bifunction in order to conduct further advancements in this area. Let X be a Hausdorff topological vector space, and Y be a real-ordered Hausdorff topological vector space. Let $K_{g\beta}$ be a nonempty general biconvex subset of X , $g: K_{g\beta} \rightarrow K_{g\beta}$ and $\beta(\cdot - \cdot): K_{g\beta} \times K_{g\beta} \rightarrow \mathbb{R}$ be a mapping. Let $P: K_{g\beta} \rightarrow 2^Y$ be a point-to-set mapping such that P is a proper, closed, and general convex cone in Y with $\text{int} P \neq \emptyset$.

A nonempty subset P of Y is called a pointed and convex cone in Y , if

- (i) $P + P = P$;
- (ii) $\lambda P \subseteq P, \forall \lambda > 0$ and
- (iii) $P \cap \{-P\} = \{0\}$.

This cone P induces a partial order on Y [22], defined as follows: For all $x, y \in Y$, we have

$$\begin{aligned} x < y &\Leftrightarrow y - x \in \text{int}P; \\ x \leq y &\Leftrightarrow y - x \in P; \\ x \nabla y &\Leftrightarrow y - x \nabla \text{int}P; \\ x \not\leq y &\Leftrightarrow y - x \notin P. \end{aligned}$$

In this research, we examine the following general vector equilibrium problem (for short, *biGVEP*) and bigeneral strong vector equilibrium problem (for short, *biGSVEP*).

Definition 1.1. A nonempty subset $K_{g\beta}$ of X is termed as general biconvex with regard to an arbitrary function $g: K_{g\beta} \rightarrow K_{g\beta}$ and bifunction $\beta(\cdot - \cdot)$, if

$$g(x) + \lambda\beta(g(y) - g(x)) \in K_{g\beta}, \forall g(x), g(y) \in K_{g\beta}, \lambda \in [0,1].$$

A $g\beta$ -connected set is the general biconvex set $K_{g\beta}$. It should be noted that while the general convex set K_g is a general biconvex set with $\beta(g(y) - g(x)) = g(y) - g(x)$ and the converse is not true.

Remark 1.2. The set $K_{g\beta} = \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$ is a general biconvex set with respect to function β , where

$$\beta(g(y) - g(x)) = \begin{cases} g(y) - g(x), & \text{for } g(y) > 0, g(x) > 0 \text{ or } g(y) < 0, g(x) < 0 \\ g(x) - g(y), & \text{for } g(y) < 0, g(x) > 0 \text{ or } g(y) < 0, g(x) < 0 \end{cases}$$

Clearly, $K_{g\beta}$ is not a general convex set.

If we take $g = I$, the identity operator, then Definition 2.1 reduces to

Definition 1.3. A set K_β is said to be biconvex with respect to arbitrary bifunction $\beta(\cdot - \cdot)$, if $u + \lambda\beta(v - u) \in K_\beta, \lambda \in [0,1]$.

The biconvex set K_β is also called a β -connected set, which was mainly introduced and studied by Noor et al. [25]. We would like to mention that the β -biconvex set is quite different from the invex considered in [1].

$$(biGVEP) \text{ Find } g(x) \in K_{g\beta} \text{ such that for all } g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0,1], \tag{1}$$

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}$$

$$(biGSVEP) \text{ Find } g(x) \in K_{g\beta} \text{ such that for all } g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0,1], \tag{2}$$

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}, \forall g(y) \in K_{g\beta},$$

Where $f, h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ are the given bifunctions.

Special Cases:

- (i) If $\beta(g(x) - g(z)) = (g(x) - g(z))$, and $\lambda_2 = 1$, then (biGVEP) reduces to the problem of determining $g(x) \in K_{g\beta}$ such that for all $g(z) \in K_{g\beta}, \lambda_1 \in (0,1]$, we have

$$f(g(z) + \lambda_1(g(x) - g(z)), g(y)) + h(g(x), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

For $g = I$, the problem (3) is studied by Ahmad and Akram [3].

- (ii) If $\lambda_1, \lambda_2 = 1$, then (biGVEP) reduces to the problem of finding $g(x) \in K_{g\beta}$ such that

$$f(g(x), g(y)) + h(g(x), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

For $g = I$, the problem (4) is called the vector equilibrium problem, introduced and studied by K. R. Kazmi [17].

(iii) If $\lambda_1, \lambda_2 = 1, P = \mathbb{R}_+, Y = \mathbb{R}$ and $\beta(g(x) - g(z)) = (g(x) - g(z))$, then (biGVEP) reduces to the problem:

$$\text{Find } g(x) \in K_{g\beta} \text{ such that} \\ f(g(x), g(y)) + h(g(x), g(y)) \geq 0, \forall g(y) \in K_{g\beta}. \tag{5}$$

It is evident that many of the current models of vector equilibrium, equilibrium problems, and related problems can be generated with a suitable choice of operators used in the formulation of (1) and (2). In this paper, bigeneral vector equilibrium problems (biGVEP) and bigeneral strong vector equilibrium problems (biGSVEP) in Hausdorff topological vector spaces are studied, and existence results are established. Our findings can be viewed as an expansion of those of Ahmad and Rahaman [23] and Noor et al. [24]. And for other applications and recent advancements, we see [2, 9, 10, 16, 17, [20, [21] and the references therein.

The structure of the paper is as follows:

Section 2 covers essential definitions and preliminary results. Section 3 presents the main theorem of the study, along with an example.

2. Preliminaries

The concepts and definitions listed below are necessary to support the results obtained in this paper.

Definition 2.1. Let $K_{g\beta}$ be a nonempty subset of a Hausdorff topological vector space X . A set-valued mapping $F: K_{g\beta} \rightarrow 2^Y$ is said to be bigeneral KKM -mapping if, for each finite subset $\{g(x_1), \dots, g(x_n)\}$ of $K_{g\beta}$, $\text{co}\{g(x_1), \dots, g(x_n)\} \subseteq \bigcup_{i=1}^n F(g(x_i))$, where $\text{co}\{g(x_1), \dots, g(x_n)\}$ denotes the convex hull of $\{g(x_1), \dots, g(x_n)\}$. The following result is a generalization of the aforementioned KKM -Theorem by K. Fan [14], and its proof can be easily extended to the mapping F and g .

Theorem 2.2. Let $K_{g\beta}$ be a subset of a topological vector space $X, g: K_{g\beta} \rightarrow K_{g\beta}$ be a mapping, and let $F: K_{g\beta} \rightarrow 2^Y$ be a bigeneral KKM -mapping. If for each $g(x) \in K_{g\beta}, F(g(x))$ is closed, and if for at least one point $g(x) \in K_{g\beta}, F(g(x))$ is compact, then $\bigcap_{g(x) \in K_{g\beta}} F(g(x)) \neq \phi$.

Definition 2.3. A mapping $h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ is called bigeneral P -monotone if and only if

$$h(g(x), g(y)) + h(g(y), g(x)) \in -P, \forall g(x), g(y) \in K_{g\beta}.$$

Definition 2.4. A mapping $h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ is called bigeneral P -monotone with respect to the line-segment $[\cdot, g(z)]$ if and only if, for each $g(z) \in K_{g\beta}, \lambda \in (0, 1]$, we have

$$h(g(z) + \lambda\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda\beta(g(y) - g(z)), g(x)) \in -P, \forall g(x), g(y) \in K_{g\beta},$$

where $[\cdot, g(z)]$ denotes the line segment joining the points, excluding the endpoint $g(z)$.

Remark 2.5. If $\lambda = 1$ and $\beta(g(x) - g(z)) = (g(x) - g(z))$, Then Definition 2.4 reduces to Definition 2.3.

Definition 2.6. Let (Y, P) be an ordered topological vector space. A mapping $T: X \rightarrow Y$ is called general P -biconvex if and only if for each pair $g(x), g(y) \in K_{g\beta}$ and $\lambda \in [0, 1]$,

$$T(g(y) + \lambda\beta(g(x) - g(y))) \leq_P \lambda T(g(x)) + (1 - \lambda)T(g(y)).$$

Lemma 2.7. [11]. Let (Y, P) be an ordered topological vector space with a pointed, closed and convex cone P . Then for all $g(x), g(y) \in Y$, we have

- (i) $g(y) - g(x) \in \text{int}P$ and $g(y) \notin \text{int}P$ imply $g(x) \notin \text{int}P$;
- (ii) $g(y) - g(x) \in P$ and $g(y) \notin \text{int}P$ imply $g(x) \notin \text{int}P$;
- (iii) $g(y) - g(x) \in -\text{int}P$ and $g(y) \notin -\text{int}P$ imply $g(x) \notin -\text{int}P$;
- (iv) $g(y) - g(x) \in -P$ and $g(y) \notin -\text{int}P$ imply $g(x) \notin -\text{int}P$;

Proof.

(i) Assume that $g(y) - g(x) \in \text{int}P$ and $g(y) \notin \text{int}P$.

We have to show that $g(x) \notin \text{int}P$. Suppose, on contradiction, $g(x) \in \text{int}P$. Then, $g(y) = g(x) + g(y) - g(x)$, where $g(x) \in \text{int}P$ and $g(y) - g(x) \in \text{int}P$. Since the interior of the convex cone is closed under addition, $g(y) \in \text{int}P$, which is a contradiction.

Hence, $g(x) \notin \text{int}P$.

(ii) Assume that $g(y) - g(x) \in P$ and $g(y) \notin \text{int}P$.

We have to show that $g(x) \notin \text{int}P$. Suppose, on contradiction, $g(x) \in \text{int}P$. Then, $g(y) = g(x) + g(y) - g(x)$, where $g(x) \in \text{int}P$ and $g(y) - g(x) \in \text{int}P$. Since the interior of the convex cone is closed under addition, it implies $g(y) \in \text{int}P$, which is a contradiction.

Hence, $g(x) \notin \text{int}P$.

(iii) Assume that $g(y) - g(x) \in -\text{int}P$ and $g(y) \notin -\text{int}P$.

We have to show that $g(x) \notin -\text{int}P$. Suppose, on contradiction, $g(x) \in -\text{int}P$. Then, $g(y) = g(x) + g(y) - g(x)$, where $g(x) \in -\text{int}P$ and $g(y) - g(x) \in -\text{int}P$.

Since $-\text{int}P$ is also convex, it implies $g(y) \in -\text{int}P$, which is a contradiction.

Hence, $g(x) \notin -\text{int}P$.

(iv) Assume that $g(y) - g(x) \in -P$ and $g(y) \notin -\text{int}P$.

We have to show that $g(x) \notin -\text{int}P$. Suppose, on contradiction that $g(x) \in -\text{int}P$. Then, $g(y) = g(x) + g(y) - g(x)$, where $g(x) \in -\text{int}P$ and $g(y) - g(x) \in -\text{int}P$. Since $-\text{int}P$ is also convex, $g(y) \in -\text{int}P$, which is a contradiction.

Hence, $g(x) \notin -\text{int}P$.

Lemma 2.8. Let X be a Hausdorff topological space, and let A_1, A_2, \dots, A_n be nonempty compact subsets of X . Then $\text{co}(\cup_{i=1}^n A_i)$ is compact.

Example 2.9. Let $X = \mathbb{R}, Y = \mathbb{R}^2, K_{g\beta} = \mathbb{R} \setminus \{0\}$,

$P = \{(g(x), g(y)) : g(x) \leq 0, g(y) \leq 0\} \subseteq Y$ and let $h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ be defined as

$$h(g(x), g(y)) = ((g(x))^2 - 2g(y) + 1, (g(x))^2 - (g(y))^2), \forall g(x), g(y) \in K_{g\beta}.$$

Now,

$$\begin{aligned} h(g(x), g(y)) + h(g(y), g(x)) &= ((g(x))^2 - 2g(y) + 1, (g(x))^2 - (g(y))^2) \\ &\quad + ((g(y))^2 - 2g(x) + 1, (g(y))^2 - (g(x))^2) \\ &= (((g(x))^2 - 2g(x) + 1) + ((g(y))^2 - 2g(y) + 1), 0) \\ &= ((g(x) - 1)^2 + (g(y) - 1)^2, 0) \in -P. \end{aligned}$$

This shows that h is bigeneral P -monotone.

Example 2.10. Let $X = Y = \mathbb{R}, K_{g\beta} = \mathbb{R} \setminus \{0\}$,

$P = \{(g(x), g(y)) : g(x) \leq 0\}$ and let $h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ be defined for each $g(z) \in K_{g\beta}$ and $\lambda \in (0, 1]$ by $h(g(z) + \lambda\beta(g(x) - g(z)), g(y)) = g(z) + \lambda(g(x) - g(z)) - \lambda g(y), \forall g(x), g(y) \in K_{g\beta}$.

Now,

$$\begin{aligned} h(g(z) + \lambda\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda\beta(g(y) - g(z)), g(x)) \\ = g(z) + \lambda(g(x) - g(z)) - \lambda g(y) + g(z) + \lambda(g(y) - g(z)) - \lambda g(x) \\ = 2(1 - \lambda)g(z) \in -P. \end{aligned}$$

This shows that h is bigeneral P -monotone with respect to the line segment $[\cdot, g(z)]$.

Remark 2.11. If $\lambda = 1$ in Example 2.10, then it can be easily verified that h is bigeneral P -monotone. as mentioned in Definition 2.3.

Example 2.12. Let $X = Y = \mathbb{R}, K_{g\beta} = \mathbb{R} \setminus \{0\}, P = \{x \in \mathbb{R} : g(x) \geq 0\}$ and $T: K_{g\beta} \times K_{g\beta} \rightarrow Y$ be defined by

$$T(g(x), g(y)) = (g(x) + g(y))^2 - 2(g(x) + g(y)), \forall g(x), g(y) \in K_{g\beta}.$$

Then, for $\lambda \in (0, 1), g(x_1), g(x_2) \in K_{g\beta}$, we have

$$\begin{aligned} T(\lambda g(x_1) + (1 - \lambda)g(x_2), g(y)) &= (\lambda g(x_1) + (1 - \lambda)g(x_2) + g(y))^2 \\ &\quad - 2(\lambda g(x_1) + (1 - \lambda)g(x_2) + g(y)). \end{aligned} \tag{6}$$

$$\lambda T(g(x_1), g(y)) = \lambda(g(x_1) + g(y))^2 - 2\lambda(g(x_1) + g(y)). \tag{7}$$

$$\begin{aligned} (1 - \lambda)T(g(x_2), g(y)) &= (1 - \lambda)(g(x_2) + g(y))^2 \\ &\quad - 2(1 - \lambda)(g(x_2) + g(y)). \end{aligned} \tag{8}$$

Adding (7) and (8), and then subtracting from (6), it can be easily verified that

$$\begin{aligned} \lambda T(g(x_1), g(y)) + (1 - \lambda)T(g(x_2), g(y)) - T(\lambda g(x_1) + (1 - \lambda)g(x_2), g(y)) \\ = \lambda(1 - \lambda)(g(x_1) - g(x_2)) \in P \end{aligned}$$

Hence,

$$T(\lambda g(x_1) + (1 - \lambda)g(x_2), g(y)) \leq_P \lambda T(g(x_1), g(y)) + (1 - \lambda)T(g(x_2), g(y)).$$

This shows that T is general P -biconvex.

3. Existence theory

First, we establish the existence of the bigeneral vector equilibrium problem (*biGVEP*).

Theorem 3.1. Let X be a Hausdorff topological vector space and Y be a real ordered Hausdorff topological vector

space, and let $K_{g\beta} \subset X$ be a nonempty, closed, and general biconvex set. Let P be a proper, pointed, closed, and general biconvex cone in Y with $\text{int}P \neq \emptyset$, and let $f, h : K_{g\beta} \times K_{g\beta} \rightarrow Y$ be the given bifunctions. Let $W : K_{g\beta} \rightarrow 2Y$ be a mapping with $W = Y \setminus -\text{int}P$ such that the following conditions hold:

- (i) f is continuous in the first variable, affine and general P -biconvex in the second variable, and $f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)), \forall g(x), g(z) \in K_{g\beta}$ and $\lambda_1 \in (0, 1]$,
- (ii) h is bigeneral P -monotone with respect to the line-segment $[\cdot, g(z))$, hemicontinuous, continuous and general P biconvex in the second variable, affine in the first variable, and $h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)), \forall g(x), g(z) \in K_{g\beta}, \lambda_2 \in (0, 1]$,
- (iii) the mapping W is closed,
- (iv) Suppose that there exists a nonempty, compact, and general biconvex subset $D_{g\beta}$ of $K_{g\beta}$ such that for each $g(x) \in K_{g\beta} \setminus D_{g\beta}$ and for each $g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1)$, there exists $g(y) \in D_{g\beta}$ such that $f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)) \notin -\text{int}P$.

Then, (biGVEP) is solvable.

We require the following two propositions for the proof of Theorem 3.1, using the same assumptions as in Theorem 3.1.

Proposition 3.2. Let $D_{g\beta} \subset K_{g\beta} \subset X$ be a nonempty compact and general biconvex set. Then there exists $g(x) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

Proof.

For each $g(y), g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1]$, consider the set

$$M(g(y)) = \{g(x) \in D_{g\beta} : f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P\}.$$

We claim that M is a KKM -mapping. Assume on the contrary that M is not a KKM -mapping, then there exists some $g(x) \in \text{co}\{g(y_1), \dots, g(y_n)\}$ such that for all $t_i \in [0, 1], i = 1, 2, \dots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$g(x) = \sum_{i=1}^n t_i g(y_i) \notin \cup_{i=1}^n M(g(y_i)).$$

Thus, we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y_i)) - h(g(z) + \lambda_2\beta(g(y_i) - g(z)), g(x)) \in -\text{int}P \tag{9}$$

and

$$\sum_{i=1}^n t_i f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y_i)) - \sum_{i=1}^n t_i h(g(z) + \lambda_2\beta(g(y_i) - g(z)), g(x)) \in -\text{int}P. \tag{10}$$

Since f is affine in the second variable and h is affine in the first variable, and using the conditions that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) = 0 = h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)),$$

we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), \sum_{i=1}^n t_i g(y_i)) - h(g(z) + \lambda_2\beta(\sum_{i=1}^n t_i g(y_i) - g(z)), g(x)) \in -\text{int}P, \tag{11}$$

which implies that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) - h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)) \in -\text{int}P. \tag{12}$$

It follows that $0 \in -\text{int}P$, which contradicts the pointedness of P and hence M is a KKM -mapping. Next, we show that $M(g(y))$ is closed. In fact, let $\{g(x_n)\}$ be a net in $M(g(y))$ such that $g(x_n) \rightarrow g(x)$. As f is continuous in the first variable and h is continuous in the second variable, we have

$$f(g(z) + \lambda_1\beta(g(x_n) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x_n))$$

converges to

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)).$$

As W is closed, we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \in W. \tag{13}$$

Thus, we obtain

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P \tag{14}$$

and so clearly $g(x) \in M(g(y))$ and $M(g(y))$ is closed.

As $M(g(y))$ is a closed subset of a compact set $D_{g\beta}$, thus $M(g(y))$ is compact. By employing the KKM -Theorem 2.2, we have

$$\cap_{g(y) \in K_{g\beta}} M(g(y)) \neq \emptyset.$$

Thus, there exists at least one $g(x) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P.$$

Proposition 3.3. The following statements are equivalent:

- (I) $g(x) \in D_{g\beta} : f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}$.

$$(II) \quad g(x) \in D_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -intP, \forall g(y) \in K_{g\beta}.$$

Proof.

Suppose that (II) holds. Then there exists $g(x) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -intP. \tag{15}$$

Since h is general P -monotone with respect to the line segment $[\cdot, g(z)]$, we have

$$h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \leq_P h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)). \tag{16}$$

Adding $f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y))$ on both sides of (16), we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \leq_P f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)). \tag{17}$$

Since $f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -intP$, using (iv) of Lemma 2.7 and (17), it follows that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -intP,$$

That is, (I) holds.

Conversely, suppose that (I) holds. Then

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)). \tag{18}$$

Let $\alpha \in (0, 1]$ for all $g(y) \in D_{g\beta}, g(x_\alpha) = g(x) + \alpha\beta(g(y) - g(x))$.

Then $g(x_\alpha) \in D_{g\beta}$, we obtain

$$(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) - (1 - \alpha)h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x)) \notin -intP. \tag{19}$$

Since h is general P -biconvex in the second variable, and using the assumption $h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)) = 0$, we obtain

$$\begin{aligned} 0 &= h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x_\alpha)) \\ &= h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x) + \alpha\beta(g(y) - g(x))) \\ &= h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x) + \alpha(g(y) - g(x))) \\ &\quad (\text{for } \beta(g(y) - g(x)) = (g(y) - g(x))) \\ &\leq_P \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)) \\ &\quad - (1 - \alpha)h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x)). \end{aligned}$$

It implies that

$$-(1 - \alpha)h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x)) \leq_P \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)). \tag{20}$$

Adding $(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha))$ on both sides of (20), we have

$$\begin{aligned} (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) - (1 - \alpha)h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(x)) \\ \leq_P (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) + \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)). \end{aligned} \tag{21}$$

Using (19), (21), and (iv) of Lemma 2.7, we obtain

$$(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) + \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)) \notin -intP. \tag{22}$$

Since f is general P -biconvex in the second variable and using the assumption

$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) = 0$, and (22), we have

$$\begin{aligned} (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) &= (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x) + \alpha\beta(g(y) - g(x))) \\ &= (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x) + \alpha(g(y) - g(x))) \\ &\quad (\text{for } \beta(g(y) - g(x)) = (g(y) - g(x))) \\ &\leq_P \alpha(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) \\ &\quad + (1 - \alpha)(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) \\ &= \alpha(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)). \end{aligned}$$

On adding $\alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y))$ to (23), it implies that (23)

$$\begin{aligned} \alpha(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)) \\ \geq (1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x_\alpha)) \notin -intP. \end{aligned}$$

Again, using (iv) of Lemma 2.7, we have

$$\alpha(1 - \alpha)f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + \alpha h(g(z) + \lambda_2\beta(g(x_\alpha) - g(z)), g(y)). \tag{24}$$

Dividing (24) by α and using the hemicontinuity of h in the first variable and closedness of W , we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \in W$$

and thus, we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -intP, \forall g(y) \in K_{g\beta}.$$

Hence, (II) holds.

Proof of Theorem 3.1. Let $\{g(y_1), g(y_2), \dots, g(y_n)\}$ be a finite subset of $K_{g\beta}$ and

$$B_{g\beta} = \text{co}(D_{g\beta} \cup \{g(y_1), g(y_2), \dots, g(y_n)\}).$$

It is clear that $B_{g\beta}$ is a compact and general biconvex subset of $K_{g\beta}$.

By Proposition 3.2, there exists $g(x) \in B_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

In particular

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y_i)) - h(g(z) + \lambda_2\beta(g(y_i) - g(z)), g(x)) \notin -\text{int}P, i = 1, 2, \dots, n.$$

Hence, every finite subfamily of closed sets

$$L(g(y)) = \{g(x) \in B_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) - h(g(z) + \lambda_2\beta(g(y) - g(z)), g(x)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}\}$$

has a nonempty intersection. Since $B_{g\beta}$ is compact, we have

$$\bigcap_{g(y) \in K_{g\beta}} L(g(y)) \neq \phi.$$

By Proposition 3.3, we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

Thus, there exists $g(x) \in B_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

By assumption (iv), there exists $g(x) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -\text{int}P, \forall g(y) \in K_{g\beta}.$$

Hence, (biGVEP) admits a solution.

Now, we establish the solvability of the bigeneral strong vector equilibrium problem (biGSVEP) under appropriate conditions.

Theorem 3.4. Let X be a Hausdorff topological vector space and Y be real ordered Hausdorff topological vector spaces and $K_{g\beta} \subset X$ be a nonempty, closed, and general biconvex set. Let $P: K_{g\beta} \rightarrow 2^Y$ be a point-to-set mapping such that P is a proper, pointed, closed, and general biconvex cone in Y with $\text{int} P \neq \phi$. Let $f, h: K_{g\beta} \times K_{g\beta} \rightarrow Y$ be the mappings such that f is affine in the second variable and h is affine in the first variable and $f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) = 0 = h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)), \forall g(x) \in K_{g\beta}$.

Let the following conditions hold:

(i) for each $g(z), g(y) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1]$, the set

$$\{g(x) \in K_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \in -P \setminus \{0\}\}$$

is open in $K_{g\beta}$;

(ii) there exists a nonempty compact and general biconvex set $D_{g\beta}$ of $K_{g\beta}$, and for each $g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1], g(x) \in K_{g\beta} \setminus D_{g\beta}$, there exists $g(u) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(u)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(u)) \in -P \setminus \{0\}.$$

Then, for each $g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1]$, there exists $g(x) \in K_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}, \forall g(y) \in K_{g\beta}.$$

Proof.

For each $g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1]$,

$$G(g(y)) = \{g(x) \in D_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}\}, \forall g(y) \in K_{g\beta}.$$

Clearly, for each $g(y) \in K_{g\beta}$

$$G(g(y)) = \{g(x) \in K_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}\} \cap D_{g\beta}, \forall g(y) \in K_{g\beta}.$$

Since $G(g(y))$ is closed subset of $D_{g\beta}$, so $G(g(y))$ is compactly closed. Let $B_{g\beta} = \text{co}(D_{g\beta} \cup \{g(y_1), g(y_2), \dots, g(y_n)\})$ for a finite subset $\{g(y_1), g(y_2), \dots, g(y_n)\}$ of $K_{g\beta}$. Then by Lemma 2.8, $B_{g\beta}$ is a compact general biconvex subset of $K_{g\beta}$.

Consider the set

$$H(g(y)) = \{g(x) \in B_{g\beta}: f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}\}, \forall g(y) \in B_{g\beta}.$$

We claim that H is a KKM -mapping. Suppose that H is not a KKM -mapping, then there exists some $g(x) \in \text{co}\{g(y_1), g(y_2), \dots, g(y_n)\}$ such that for all $t_i \in [0, 1], i = 1, 2, \dots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$g(x) = \sum_{i=1}^n t_i g(y_i) \notin \bigcup_{i=1}^n H(g(y_i)).$$

That is,

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y_i)) + h(g(z) + \lambda_2\beta(g(y_i) - g(z)), g(x)) \in -P \setminus \{0\}, \tag{25}$$

which implies that

$$\sum_{i=1}^n t_i f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y_i)) + \sum_{i=1}^n t_i h(g(z) + \lambda_2\beta(g(y_i) - g(z)), g(x)) \in -P \setminus \{0\}. \tag{26}$$

Since h and f are affine in the first and second variable respectively, and from (26), we have

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)) \in -P \setminus \{0\}. \tag{27}$$

But

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(x)) = 0 = h(g(z) + \lambda_2\beta(g(x) - g(z)), g(x)).$$

From (27), we have $0 \in -P \setminus \{0\}$, which is not possible.

Hence H is a *KKM* -mapping.

Since $H(g(y))$ is a closed subset of $B_{g\beta}$ and thus compact. Hence by Theorem 2.2, we have

$$\bigcap_{g(y) \in B_{g\beta}} H(g(y)) \neq \phi. \tag{28}$$

Thus, there exists

$$g(y_0) \in \bigcap_{g(y) \in B_{g\beta}} H(g(y)).$$

Next, we show that $g(y_0) \in D_{g\beta}$. In fact, if $g(y_0) \in B_{g\beta} \setminus D_{g\beta}$, then by condition (ii), there exists $g(u_0) \in D_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(y_0) - g(z)), g(u_0)) + h(g(z) + \lambda_2\beta(g(y_0) - g(z)), g(u_0)) \in -P \setminus \{0\}, \tag{29}$$

which contradicts $g(y_0) \in H(g(y))$ and so $g(y_0) \in D_{g\beta}$.

Since $G(g(y_i)) = H(g(y_i)) \cap D_{g\beta}$, for each $g(y_i) \in B_{g\beta}, i = 1, 2, \dots, n$, we have

$$g(y_0) \in \bigcap_{i=1}^n G(g(y_i)).$$

That is, $\bigcap_{i=1}^n G(g(y_i)) \neq \phi$, for each $g(y_i) \in K_{g\beta}$.

By the compactness of $G(g(y))$, we see, for each $g(y) \in K_{g\beta}$, that there exists $g(x_0) \in D_{g\beta}$ such that $g(x_0) \in$

$$\bigcap_{g(y) \in K_{g\beta}} G(g(y)) \neq \phi.$$

Therefore, for each $g(z) \in K_{g\beta}, \lambda_1, \lambda_2 \in (0, 1]$, there exists $g(x) \in K_{g\beta}$ such that

$$f(g(z) + \lambda_1\beta(g(x) - g(z)), g(y)) + h(g(z) + \lambda_2\beta(g(x) - g(z)), g(y)) \notin -P \setminus \{0\}, \forall g(y) \in K_{g\beta}.$$

Thus, (biGSVEP) is solvable.

4. Conclusion

The paper develops a unified and generalized framework for studying vector equilibrium problems in Hausdorff topological vector spaces by introducing the notions of general biconvex sets and functions defined via an arbitrary mapping and bifunction. Within this framework, the authors formulate both a bigeneral vector equilibrium problem (biGVEP) and a bigeneral strong vector equilibrium problem (biGSVEP), showing that many classical models, such as equilibrium problems, vector equilibrium problems, and related variational inequalities, arise as special cases under suitable choices of parameters and mappings. Using generalized KKM techniques, along with properties like P - monotonicity, generalized convexity, and compactness, the paper establishes sufficient conditions for the existence of solutions to both (biGVEP) and (biGSVEP). Supporting propositions and examples illustrate the applicability of the assumptions and highlight the role of the introduced generalized structures. Overall, the results extend several well-known existence theorems in the literature and provide a broader setting that can accommodate nonconvex and more complex equilibrium models, thereby opening avenues for further research in generalized equilibrium theory. It is expected that the approaches, concepts, and techniques used in this publication will encourage additional research in this area.

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