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PARAMETER ESTIMATION FOR A CLASS OF DIFFUSIONPROCESS FROM DISCRETE OBSERVATION

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Abstract:-

This paper is concerned with the parameter estimation problem for a class of diffusion process from discrete observations. The approximate likelihood function is given by using a Riemann sum and an Itˆo sum to approximate the integrals in the continuous-time likelihood function. The consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation are proved by applying the martingale moment inequality, Holder's inequality, Chebyshev inequality, B-D-G inequality and uniform ergodic theorem. The results are applied to the hyperbolic process.

Keywords:-*Diffusion process, parameter estimation, discrete observation, consistency, asymptotic normality.*

I. INTRODUCTION

Stochastic phenomenon occurs in many fields such as biology and medical science and stochastic models have come to play an important role in many branches of science and industry where more and more people have encountered stochastic process. Stochastic processes are of great importance on studying the stochastic phenomena and have been widely employed for model building in statistical physics, management decisions, economic mathematics, computer science, etc. Recently, stochastic processes, especially diffusion processes defined by stochastic differential equations, have been widely used to describe the price dynamics of a financial asset such as interest rate, discount bonds and futures, (see e.g. [15, 17]). The Black-Scholes option pricing model described by a geometric Brownian motion (see e.g. [5]) and the Vasicek and Cox-Ingersoll-Ross models developed based on two specific mean-reverting diffusion processes (see e.g. [7,8,20]) are widely used models in economic cases. Therefore, diffusion processes are the basic stochastic modeling instruments in the modern financial theory and applications of diffusion processes have been more and more popular. However, in the actual systems described by the stochastic models, part or all of the parameters are always unknown, but the observed values are known. Hence, the unknown parameters are needed to be estimated based on observations. In terms of the specific stochastic models, it is necessary to estimate the parameters in these models to obtain proper structures no matter what methods are used.

In earlier works, some methods have been considered to estimate the parameters in diffusion processes based on continuous-time observations. For example, maximum likelihood estimation for scalar parameters

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And vector parameters in nonlinear stochastic differential equations (see e.g. [2,16,21,22]), Bayes estimation for a certain class of diffusion processes (see e.g. [4]) and other methods such as minimum contrast method (see e.g. [14]), M-estimation method (see e.g. [23,24]) and sequential estimation method (see e.g. [9]). However, in fact, it is impossible to observe a process continuously in time. Therefore, parametric inference based on sampled data is important in dealing with practical problems. Some methods have been applied to solve the parameter estimation problem for the diffusion processes based on discrete observations. For example, Kessler [11] obtained the contrast function by discretizing the process based on Euler method and considering Gaussian approximation to the transition density, Yoshida [25] and Bibby [3] derived the approximate likelihood function and martingale estimation function respectively by using Itˆo sum and Riemann sum to approximate the integrals in the continuous-time likelihood function, A¨ıt-Sahalia [1] and Chang [6] obtained the approximate transition densities based on series expansions, Uchida [19] considered adaptive maximum likelihood type estimation of both drift and diffusion parameters for an ergodic diffusion process. When the process is observed partially, Kristensen [12] and Gu [10] applied extended Kalman filter and local linearization filter respectively to estimate both the states and parameters.

In our work, parameter estimation problem for a class of ergodic diffusion process is studied from discrete observations. Although the problem has been discussed in some literatures such as [13] and [18], the asymptotic normality of the error of estimation has not been discussed. In this paper, the process, conditions and methods are different from that in [13] and [18]. Firstly, the approximate likelihood function is given by using Itˆo sum and Riemann sum to replace the integrals in the continuous-time likelihood function. Then, the limitation of the approximate likelihood function is obtained and this limitation is assumed to attain the maximum at the true parameter value. Hence, the consistency of the maximum likelihood estimator is proved. The asymptotic normality of the error of estimation is proved by applying the Holder's inequality, B-D-G inequality, Chebyshev inequality, dominated convergence theorem and uniform ergodic theorem. Finally, the Ornstein-Uhlenbeck process is introduced as an example to verify the results.

This paper is organized as follows. In Section 2 some assumptions and the approximate likelihood function are given. Some lemmas and the main results are given in Section 3. In this section, the consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation are proved. In Section 4 the hyperbolic process is introduced as an example. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

We study a class of one-dimensional stationary ergodic diffusion processes described by the following stochastic differential equation:

$$\begin{cases} dX_t = f(X_t,\theta)dt + g(X_t)dW_t \\ X_0 = x_0, \end{cases} \quad (1)$$

where $\theta \in \Theta$ a compact subset in R is an unknown one-dimensional parameter and $(W_t, t \geq 0)$ is a standard Wiener process defined on a complete probability space (Ω, F, P). The drift and the diffusion coefficient are supposed to be known and do not depend on the time *t*.

The process is observed at discrete equidistant times $t_i = i\Delta, 1 \leq i \leq n, \Delta > 0$ and the parameter is estimated based on the discrete data. The approximate likelihood function is given by using a Riemann sum and an Itˆo sum to approximate the integrals in the continuous-time likelihood function. From now on, we will work under the assumptions as follows:

Assumption 1: $|f(x,\theta)| + |g(x)| \leq K_1(1 + |x|)$ and $|f(x,\theta) - f(y,\theta)| + |g(x) - g(y)| \leq K_2|x - y|$ where K_1 and K_2 are constants and $x,y \in$ R.

Assumption 2: $|f^{\theta}(x,\theta) - f^{\theta}(y,\theta)| \leq K_3|x - y|$ where f^{θ} denotes the first differential with respect to θ, K_3 is a constant and $x,y \in \mathbb{R}$.
Assumption 3: $\mathrm{E}|X_0|^p < \infty$ and $\inf_x g^2(x) > 0$ for each $p > 0$.
Assumption 4: $|f^{\theta}(x,s)| < h(x)$ and $|f^{\theta\theta}(x,s)| < m(x)$ for all $s \in I(\theta)$ where $I(\theta)$ is a closed interval containing θ and $f^{\theta\theta}$ denotes the second differential with respect to θ. Moreover, $\mathbb{E}[\frac{h(x_0)}{g(x_0)}]^2 < \infty$ and $\mathbb{E}[\frac{m(x_0)}{g(x_0)}]^2 < \infty$.

Assumption 5: The function $\mathbb{E}[\frac{f(x_0,\theta)(f(x_0,\theta_0) - \frac{1}{2}f(x_0,\theta)}{g^2(x_0)}]$ has its unique maximum at the true parameter value θ_0.

Let P_{θ}^t be the probability measure generated by the process $\{X_s, 0 \leq s \leq t\}$ and P_W^t be the probability measure induced by the standard Wiener process. The continuous-time log-likelihood function has the following expression

$$\ell_t(\theta) = \log \frac{dP_{\theta}^t}{dP_W^t} = \int_0^t \frac{f(X_s,\theta)}{g^2(X_s)} dX_s - \frac{1}{2}\int_0^t \frac{f^2(X_s,\theta)}{g^2(X_s)} ds. \quad (2)$$

Then, the approximate likelihood function can be written as

$$\ell_n(\theta) = \sum_{i=1}^{n} \frac{f(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(X_{t_i} - X_{t_{i-1}}) - \frac{\Delta}{2}\sum_{i=1}^{n} \frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})} . \quad (3)$$

III. Main Results and Proofs

The following lemmas are useful to derive our results.

Lemma 1: Assume that $\{X_t\}$ is a solution of the stochastic differential equation 1 and Assumptions 1-3 hold. Then, for any integer $n \geq 1$ and $0 \leq s \leq t$,

$$\mathrm{E}|X_t - X_s|^{2p} = O(|t - s|^p).$$

Proof: Suppose θ_0 is the true parameter value, by applying Holder's inequality, it follows that

$$\begin{aligned}
&|Xt - Xs|2p \\
=\;& |\int_s^t f(X_u,\theta_0)du + \int_s^t g(X_u)dW_u|^{2p} \\
\leq\;& 2^{2p-1}(|\int_s^t f(X_u,\theta_0)du|^{2p} + |\int_s^t g(X_u)dW_u|^{2p}) \\
\leq\;& 2^{2p-1}((t-s)^{2p-1}\int_s^t |f(X_u,\theta_0)|^{2p}du + |\int_s^t g(X_u)dW_u|^{2p})
\end{aligned}$$

Since

$$|f(X_u,\theta_0)|^{2p} \leq K_1^p 2^{p-1}(1 + |X_u|^{2p}), \quad (4)$$

from Assumption 3 together with the stationarity of the process, one has

$$\mathbb{E}[\int_s^t |f(X_u,\theta_0)|^{2p}du] = O(|t - s|) . \quad (5)$$

From B-D-G inequality, it can be checked that

$$\mathbb{E}[|\int_s^t g(X_u)dW_u|^{2p}] \le C_p \mathbb{E}|\int_s^t g^2(X_u)du|^p \quad (6)$$

where C_p is a positive constant depending only on p. Moreover,

$$\mathbb{E}|\int_s^t g^2(X_u)du|^p = O(|t-s|^p), \quad (7)$$

it follows that

$$\mathbb{E}[|\int_s^t g(X_u)dW_u|^{2p}] = O(|t-s|^p) \quad (8)$$

From the above analysis, one has

$$\mathrm{E}|X_t - X_s|^{2p} = O(|t-s|^p). \quad (9)$$

The proof is complete. ■

Lemma 2: Under Assumptions 1-3, when $\Delta \to 0$, one has

$$\mathbb{E}|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds - \sum_{i=1}^n \frac{f(X_{t_{i-1}},\theta)f(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}\Delta| \to 0.$$

Proof: By applying Holder's inequality, it can be checked that

$$\begin{aligned}
&\mathbb{E}|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds - \sum_{i=1}^n \frac{f(X_{t_{i-1}},\theta)f(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}\Delta| \\
= \;& \mathbb{E}|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds| \\
\le \;& \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E}|\frac{f(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}|ds \\
\le \;& \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\mathbb{E}[\frac{f(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}]^2)^{\frac{1}{2}} (\mathbb{E}[f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0)]^2)^{\frac{1}{2}} ds.
\end{aligned}$$

From Lemma 1 together with Assumptions 1 and 3, one has

$$\mathrm{E}[f(X_s,\theta_0) - f(X_{ti-1},\theta_0)]^2 = O(\Delta), \quad (10)$$

and $\mathbb{E}[\frac{f(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}]^2$ is bounded.

From the above analysis, we have

$$\mathbb{E}|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds - \sum_{i=1}^n \frac{f(X_{t_{i-1}},\theta)f(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}\Delta| \to 0, \quad (11)$$

as $\Delta \to 0$.

The proof is complete. ■

Lemma 3: Under Assumptions 1-3, we have

$$\mathbb{E}|\sum_{i=1}^n \frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}\Delta - \int_0^t \frac{f^2(X_s,\theta)}{g^2(X_s)}ds| \to 0,$$

as $\Delta \to 0$.

Proof: From the Holder's inequality, Assumption 1 together with the stationarity of the process, one has

$$\begin{aligned}
&\mathbb{E}|\sum_{i=1}^{n}\frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}\Delta-\int_0^t\frac{f^2(X_s,\theta)}{g^2(X_s)}ds| \\
=\ &\mathbb{E}|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[\frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}-\frac{f^2(X_s,\theta)}{g^2(X_s)}]ds| \\
=\ &\mathbb{E}|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}[\frac{f^2(X_{t_{i-1}},\theta)g^2(X_s)-f^2(X_s,\theta)g^2(X_{t_{i-1}})}{g^2(X_{t_{i-1}})g^2(X_s)}]ds| \\
=\ &\mathbb{E}|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f^2(X_{t_{i-1}},\theta)-f^2(X_s,\theta)}{g^2(X_{t_{i-1}})}ds+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f^2(X_s,\theta)(g^2(X_s)-g^2(X_{t_{i-1}}))}{g^2(X_{t_{i-1}})g^2(X_s)}ds| \\
\leq\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathbb{E}[\frac{|f(X_{t_{i-1}},\theta)+f(X_s,\theta)||f(X_{t_{i-1}},\theta)-f(X_s,\theta)|}{g^2(X_{t_{i-1}})}]ds \\
+\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathbb{E}[\frac{f^2(X_s,\theta)|g(X_s)+g(X_{t_{i-1}})||g(X_s)-g(X_{t_{i-1}})|}{g^2(X_{t_{i-1}})g^2(X_s)}]ds \\
\leq\ &K\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbb{E}|X_{t_{i-1}}-X_s|^2)^{\frac{1}{2}}ds+H\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbb{E}|X_{t_{i-1}}-X_s|^2)^{\frac{1}{2}}ds,
\end{aligned}$$

where K and H are constants.

According to the above analysis together with Lemma 1, it follows that

$$\mathbb{E}|\sum_{i=1}^{n}\frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}\Delta-\int_0^t\frac{f^2(X_s,\theta)}{g^2(X_s)}ds|\to 0,$$

as $\Delta \to 0$.

The proof is complete. ■

Remark 1: By employing the Holder's inequality, B-D-G inequality and the stationarity of the process, the above lemmas have been proved. These lemmas paly a key role in the proof of the following main results. Moreover, it is easy to check that when $f = f^0$, the results in Lemma 2 and Lemma 3 are correct as well.

In the following theorem, the consistency in probability of the maximum likelihood estimator is proved by applying martingale moment inequality, Chebyshev inequality, uniform ergodic theorem and the results of Lemmas 1-3.

Theorem 1: When $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$,

$$\hat{\theta} \xrightarrow{P} \theta_0.$$

Proof: According to the expression of the approximate likelihood function and equation (1), it follows that

$$\begin{aligned}
&\ell_n(\theta) \\
=\ &\sum_{i=1}^{n}\frac{f(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(\int_{t_{i-1}}^{t_i}f(X_s,\theta_0)ds+\int_{t_{i-1}}^{t_i}g(X_s)dW_s)-\frac{\Delta}{2}\sum_{i=1}^{n}\frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})} \\
=\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s-\frac{1}{2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}ds \\
=\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-\frac{1}{2}f(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s.
\end{aligned}$$

Then, we have

$$\frac{1}{n\Delta}\ell_n(\theta)=\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-\frac{1}{2}f(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds+\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s. \quad (12)$$

From the martingale moment inequality, it can be checked that

$$\begin{aligned}
&\mathbb{E}|\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s|^2 \\
\leq\ &\frac{1}{(n\Delta)^2}C\mathbb{E}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})})^2ds \\
\leq\ &\frac{1}{n\Delta}C_1 \\
\to\ &0,
\end{aligned}$$

where C and C_1 are constants.

By applying Chebyshev inequality, it can be obtained that

$$\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s\xrightarrow{P}0 \quad (13)$$

when $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$.

From Lemmas 2-3 together with the uniform ergodic theorem (see e.g. [16]), one has

$$\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-\frac{1}{2}f(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds \xrightarrow{p} \mathbb{E}[\frac{f(x_0,\theta)(f(x_0,\theta_0)-\frac{1}{2}f(x_0,\theta)}{g^2(x_0)}] \quad (14)$$

when $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$.
Hence, it leads to the relation that

$$\frac{1}{n\Delta}\ell_n(\theta) \xrightarrow{P} \mathbb{E}[\frac{f(x_0,\theta)(f(x_0,\theta_0)-\frac{1}{2}f(x_0,\theta)}{g^2(x_0)}] \quad (15)$$

when $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$.
From Assumption 5, it is easy to check that

$$\widehat{\theta_0} \xrightarrow{P} \theta_0, \quad (16)$$

when $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$.
The proof is complete. ■

In the following theorem, the asymptotic normality of the error of estimation is proved by employing martingale moment inequality, Chebyshev inequality, uniform ergodic theorem and the dominated convergence theorem.

Theorem 2: When $\Delta \to 0$, $n^{\frac{1}{2}}\Delta \to 0$ and $n\Delta \to \infty$ as $n \to \infty$, $\sqrt{n\Delta}(\theta_0-\widehat{\theta_0}) \xrightarrow{d} N(0, \frac{1}{\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2})$.

Proof: Expanding $\ell'_n(\theta_0)$ about θ_{b0}, it follows that

$$\ell'_n(\theta_0) = \ell'_n(\widehat{\theta_0}) + \ell''_n(\widetilde{\theta})(\theta_0-\widehat{\theta_0}), \quad (17)$$

where θ_e is between $\widehat{\theta_0}$ and θ_0.

In view of Theorem 1, it is known that $\ell'_n(\widehat{\theta_0}) = 0$, then

$$\ell'_n(\theta_0) = \ell''_n(\widetilde{\theta})(\theta_0-\widehat{\theta_0}). \quad (18)$$

Since

$$\begin{aligned}
&\ell''_n(\theta)\\
=\ & \sum_{i=1}^{n}\frac{f''(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(X_{t_i}-X_{t_{i-1}}) - \Delta\sum_{i=1}^{n}\frac{(f'^2(X_{t_{i-1}},\theta)+f(X_{t_{i-1}},\theta)f''(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}\\
=\ & \sum_{i=1}^{n}\frac{f''(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(\int_{t_{i-1}}^{t_i}f(X_s,\theta_0)ds + \int_{t_{i-1}}^{t_i}g(X_s)dW_s)\\
-\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(f'^2(X_{t_{i-1}},\theta)+f(X_{t_{i-1}},\theta)f''(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds\\
=\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s\\
-\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(f'^2(X_{t_{i-1}},\theta)+f(X_{t_{i-1}},\theta)f''(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds\\
=\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s\\
-\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'^2(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}ds,
\end{aligned}$$

we have

$$\begin{aligned}
&\ell''_n(\theta_0)\\
=\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta_0)g(X_s)}{g^2(X_{t_{i-1}})}dW_s\\
-\ & \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'^2(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}ds.
\end{aligned}$$

From the same method used in Theorem 1, it is easy to check that

$$\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta_0)g(X_s)}{g^2(X_{t_{i-1}})}dW_s \xrightarrow{P} 0 \quad (19)$$

By applying the results of Lemmas 2-3 and the uniform ergodic theorem, it follows that

$$\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'^2(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}ds\xrightarrow{p}\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2$$

$\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2>0.$

$$\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f''(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds\xrightarrow{p}0 \tag{20}$$

and

(21)

we assume that

Therefore, we have

$$\frac{1}{n\Delta}\ell''_n(\theta_0)\xrightarrow{P}-\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2 \tag{22}$$

According to the expression of $\ell''_n(\theta)$, it can be obtained that

$$\begin{aligned}
&\frac{1}{n\Delta}(\ell''_n(\widetilde{\theta})-\ell''_n(\theta_0))\\
=\ &\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(f''(X_{t_{i-1}},\widetilde{\theta})(f(X_s,\theta_0)-f(X_{t_{i-1}},\widetilde{\theta}))-f''(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0)))}{g^2(X_{t_{i-1}})}ds\\
+\ &\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(f''(X_{t_{i-1}},\widetilde{\theta})-f''(X_{t_{i-1}},\theta_0))g(X_s)}{g^2(X_{t_{i-1}})}dW_s-\frac{1}{n\Delta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(f'^2(X_{t_{i-1}},\widetilde{\theta})-f'^2(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds.
\end{aligned}$$

Then, by employing the martingale moment inequality, Chebyshev inequality, the uniform ergodic theorem and the dominated convergence theorem, it follows that

$$\frac{1}{n\Delta}(\ell''_n(\widetilde{\theta})-\ell''_n(\theta_0))\xrightarrow{P}0. \tag{23}$$

Hence, it can be obtained that

$$\frac{1}{n\Delta}\ell''_n(\widetilde{\theta})\xrightarrow{P}-\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2. \tag{24}$$

Since

$$\begin{aligned}
&\ell'_n(\theta)\\
=\ &\sum_{i=1}^{n}\frac{f'(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(X_{t_i}-X_{t_{i-1}})-\Delta\sum_{i=1}^{n}\frac{f(X_{t_{i-1}},\theta)f'(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}\\
=\ &\sum_{i=1}^{n}\frac{f'(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}(\int_{t_{i-1}}^{t_i}f(X_s,\theta_0)ds+\int_{t_{i-1}}^{t_i}g(X_s)dW_s)-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)f'(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}ds\\
=\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta)f(X_s,\theta_0)}{g^2(X_{t_{i-1}})}ds+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f(X_{t_{i-1}},\theta)f'(X_{t_{i-1}},\theta)}{g^2(X_{t_{i-1}})}ds\\
=\ &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta))}{g^2(X_{t_{i-1}})}ds+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta)g(X_s)}{g^2(X_{t_{i-1}})}dW_s,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{1}{\sqrt{n\Delta}}\ell'_n(\theta_0)\\
=\ &\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds+\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)g(X_s)}{g^2(X_{t_{i-1}})}dW_s.
\end{aligned}$$

As

$$\begin{aligned}
&\mathbb{E}|\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds|\\
\leq\ &\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathbb{E}|\frac{f'(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}|ds\\
\leq\ &\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbb{E}[\frac{f'(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}]^2)^{\frac{1}{2}}(\mathbb{E}[f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0)]^2)^{\frac{1}{2}}ds
\end{aligned}$$

it is easy to check that $\mathbb{E}[\frac{f'(X_{t_{i-1}},\theta_0)}{g^2(X_{t_{i-1}})}]^2$ is bounded.

From the Lemma 1 and Assumption 1, we have

$$\mathrm{E}[f(X_s,\theta_0) - f(X_{t_{i-1}},\theta_0)]^2 = O(\Delta). \quad (25)$$

Then, it follows that

$$\mathbb{E}|\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds| \to 0 \quad (26)$$

when $\Delta \to 0$, $n^{\frac{1}{2}}\Delta \to 0$ and $n\Delta \to \infty$ as $n \to \infty$.

By applying the Chebyshev inequality, it can be obtained that

$$\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)(f(X_s,\theta_0)-f(X_{t_{i-1}},\theta_0))}{g^2(X_{t_{i-1}})}ds \xrightarrow{P} 0 \quad .(27)$$

It is obviously that

$$\frac{1}{\sqrt{n\Delta}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f'(X_{t_{i-1}},\theta_0)g(X_s)}{g^2(X_{t_{i-1}})}dW_s \xrightarrow{d} N(0,\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2) \quad .(28)$$

Hence, we have

$$\frac{1}{\sqrt{n\Delta}}\ell_n'(\theta_0) \xrightarrow{d} N(0,\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2) \quad ,(29)$$

when $\Delta \to 0$, $n^{\frac{1}{2}}\Delta \to 0$ and $n\Delta \to \infty$ as $n \to \infty$.

From the above analysis, it can be checked that

$$\sqrt{n\Delta}(\theta_0 - \widehat{\theta}_0) \xrightarrow{d} N(0,\frac{1}{\mathbb{E}[\frac{f'(x_0,\theta_0)}{g(x_0)}]^2}) \quad ,(30)$$

when $\Delta \to 0$, $n^{\frac{1}{2}}\Delta \to 0$ and $n\Delta \to \infty$ as $n \to \infty$.

The proof is complete.

Remark 2*:* The consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation for a class of stationary ergodic diffusion processes have been proved based on discrete observations. The Lemmas 1-3 are of great importance for obtaining the main results.

IV. Example

Consider the one-dimensional hyperbolic process described by the following stochastic differential equation:

$$\begin{cases} dX_t = \theta\frac{X_t}{\sqrt{1+X_t^2}}dt + \sigma dW_t \\ X_0 \sim u_\theta, \end{cases} \quad (31)$$

where $\theta < 0, \sigma > 0$, u_θ is the invariant measure.

Firstly, it is easy to verify that this process is a stationary ergodic diffusion process.

Then, the continuous-time log likelihood function has the following expression

$$\ell_T(\theta) = \int_0^T \frac{\theta}{\sigma^2}\frac{X_t}{\sqrt{1+X_t^2}}dX_t - \frac{1}{2}\frac{\theta^2}{\sigma^2}\int_0^T\frac{X_t^2}{1+X_t^2}dt. \quad (32)$$

Hence, the approximate likelihood function is written as

$$\ell_n(\theta) = \frac{\theta}{\sigma^2}\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(X_{t_i}-X_{t_{i-1}}) - \frac{1}{2}\frac{\theta^2}{\sigma^2}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\Delta \quad .(33)$$

We obtain the expression of the maximum likelihood estimator

$$\widehat{\theta}_n = \frac{\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(X_{t_i}-X_{t_{i-1}})}{\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\Delta} \quad (34)$$

Since $\frac{|X_t|}{\sqrt{1+X_t^2}} < 1$ and $\mathbb{E}[\theta\frac{X_0}{\sqrt{1+X_0^2}}(\theta_0\frac{X_0}{\sqrt{1+X_0^2}} - \frac{1}{2}\theta\frac{X_0}{\sqrt{1+X_0^2}})]$ attains the unique maximum at $\theta = \theta_0$, it is easy to check that the coefficients of hyperbolic process satisfy the Assumptions 1-5.

Therefore, when $\Delta \to 0$, $n \to \infty$ and $n\Delta \to \infty$,

$$\widehat{\theta}_n \xrightarrow{P} \theta_0. (35)$$

When $\Delta \to 0$, $n^{\frac{1}{2}}\Delta \to 0$ and $n\Delta \to \infty$ as $n \to \infty$,

$$\sqrt{n\Delta}(\theta_0 - \widehat{\theta}_n) \xrightarrow{d} \mathbb{E}[\frac{X_0^2}{1+X_0^2}]) \quad .(36)$$

Conclusion

The current work concerns the maximum likelihood estimation of the drift parameter for a class of stationary ergodic diffusion process. The consistency of the maximum likelihood estimator has been proved by applying martingale moment inequality, Chebyshev inequality and uniform ergodic theorem. The asymptotic normality of the error of estimation has been proved by employing the Holder's inequality, B-D-G inequality, Chebyshev inequality, dominated convergence theorem and uniform ergodic theorem. hyperbolic process process has been introduced to verify the results. In practice, it is impossible to observe a process continuously over any given time period. Therefore, statistical inference based on sampled data is of major importance in dealing with practical problems. Further topics will include the multiparameter estimation for the diffusion processes from discrete observation.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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