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ORDER STATISTICS OF GEOMETRIC DISTRIBUTION

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Abstract:-
This paper mainly studies the order statistics of geometric distribution. The paper deduces the joint frequency function and conditional joint frequency function of the order statistics, and, obtain and prove some important propositions of order statistics of geometric distribution. Certain propositions are different from and also similar to corresponding propositions of exponential distribution.

Index Terms:-*Geometric distribution, order statistics, exponential distribution, joint frequency function, identical distribution*

I. INTRODUCTION

Geometric distribution has already been applied to more fields, and it has an extremely important positionespecially in some fields such as information engineering, electronic engineering, control theory and economics.It is well known that exponential distribution plays quite an important role in the statistical analysis of reliability.However in discrete life case, geometric distribution play the role of exponential distribution in continuous life case,so the study on geometric distribution becomes more and more important. [1] first proposed that the characteristicsof geometric distribution might be described by order statistics. [2] made the further study on order statistics ofgeometric distribution. [3] Obtained certain characterizations of exponential and geometric distributions. [4] studieda characterization of the geometric distribution. [5] Proved a characterization of the geometric distribution. [6] giveda note on characterizations of the geometric distribution. [7] Obtained some results for type i censored sampling fromgeometric distributions. [8] gived and proved two characterizations of geometric distributions. [9] Compared somecharacterizations of the geometric with exponential random variables. [10] Made statistical analysis for geometricdistribution based on records. [11] Got a generalization of the geometric distribution. [12] Gives a generalization ofgeometric distribution. [13] pvoved characterizations of the geometric distribution via residual lifetime.Although both geometric distribution and exponential distribution have no memory, properties of their orderstatistics make an obvious difference because of their individual differences. This paper obtains and proves somepropositions of order statistics of geometric distribution, and certain propositions are different from and also similarto corresponding propositions of exponential distribution.

II. THE RESULTS AND PROOFSA

random variableXis said to have a geometric distribution with parameterpif its frequency function isP(X=k) =pqk−1fork= 1,2,···,(1)This work was supported in part by the Key Scientific Research Program of Colleges and Universities of Henan Province of China underGrant 16A110001 and Grant 18A110009.C. He is with the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China (Email:chaobing5@163.com.)

where0< p <1andq= 1−p. We will sometimes writeX∼Geo(p).Suppose thatX1,···, Xnare i.i.d. geometric random variables. We arrangeXi's in ascending order, and someofXi's are taken as the same group whose values are equal. ThereforeXi's are divided into finite group. Then wedefineYito be the number of variables included by thei-th group andX(i)the common value of thei-th grouprandom variables with16i6n. LetDi=X (i) −X (i−1) withX (0) = 0.

Proposition 1: Based on above symbols defined, the following are consequence ofXi's:

(i) The joint frequency function ofX(1), X(2),···, X(r)is

P(X(1)=k1,X(2)=k2,···,X(r)=kr)(r∑i=1pki+ ˜pkr)n−∑16i1<i2<···<ir−16r(r−1∑j=1pkij+ ˜pkr)n−∑16i1<i2<···<ir−26r(r−2∑j=1pkij+ ˜pkr)n,(2)wherepki=P(X=ki)and˜pkr=P(X>kr), ifr=n, then˜pkr= 0, and16k16k2···6kr.

(ii) **ConditionalY1**=m1, Y2=m2,···, Yr−1=mr−1,

Dr∼Geo(1−qn−m1−···−mr−1),wherem1+m2+···+mr−16n−1.

(iii) **ConditionalY1**=m1, Y2=m2,···, Yr−1=mr−1, the frequency function ofYrisP(Yr=mr|Y1=m1,···, Yr−1=mr−1) =(n−m1−···−mr−1mr)(pq)mrqn−m1−···−mr−11−qn−m1−···−mr−1, (3)

wherem1+m2+···+mr6n.

(iv) **ConditionalY**1=m1,···, Yr−2=mr−2, Xr−1=kr−1, Yr−1=mr−1, variablesX(r)andYrare indepen-dent.

(v) The joint frequency function ofY1, Y2,···, YrisP(Y1=m1,···, Yr=mr) =n!m1!···mr+1!p∑ri=1miqn+(n−m1)+···+(n−m1−···−mr−1)−∑ri=1mi(1−qn)(1−qn−m1)···(1−qn−m1−···−mr−1),(4)especiallyP(Xi6=Xjfori6=j, i, j= 1,2,···, n) =n!pnqn(n−12n∏i=111−qi. (5)

(vi) **ConditionalY1**=m1,···, Yr=mr, variablesD1,···, Drare independent andDi∼Geo(1−qn−m1−···−mi−1), i= 1,2,···, r,especially conditionalXi6=Xj(i6=j, i, j= 1,2,···, n),Di∼Geo(1−qn−i+1), i= 1,2,···, n.Proof:(i) For convenience, we can define the eventsB=the values ofXi's are onlyk1,···, kror more thankr,Ci=at least one ofXi's takes valueki,

wherei= 1,···, N.

Set Ai=BCiwithi= 1, ···, N. It is easy to check that

Ai=the values ofXi's are onlyk1, ···, ki−1, ki+1, ···, kror more thankr.

By applying properties of probability and combinatorial arguments, it follows that

$$\begin{aligned}
&P(X_{(1)}=k_1,\cdots,X_{(r)}=k_r,N\geqslant r)\\
=\;& P(B\bigcap_{i=1}^{r}C_i)=P(B-\overline{\bigcap_{i=1}^{r}C_i})=P(B-\bigcup_{i=1}^{r}B\overline{C}_i)=P(B)-P(\bigcup_{i=1}^{r}B\overline{C}_i)=P(B)-P(\bigcup_{i=1}^{r}A_i)\\
=\;& P(B)-\sum_{i=1}^{r}P(A_i)+\sum_{1\leqslant i<j\leqslant r}^{r}P(A_iA_j)-\sum_{1\leqslant i<j<k\leqslant r}^{r}P(A_iA_jA_k)+\cdots+(-1)^rP(A_1\cdots A_r)\\
=\;& (\sum_{i=1}^{r}p_{k_i}+\tilde{p}_{k_r})^n-\sum_{1\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant r}(\sum_{j=1}^{r-1}p_{k_{i_j}}+\tilde{p}_{k_r})^n-\sum_{1\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant r}(\sum_{j=1}^{r-2}p_{k_{i_j}}+\tilde{p}_{k_r})^n.
\end{aligned}$$

(ii) The joint frequency function of $(X_{(1)},Y_1),\cdots,(X_{(r)},Y_r)$ is

$$\begin{aligned}
&P(X_{(1)}=k_1,Y_1=m_1,\cdots,X_{(r)}=k_r,Y_r=m_r,N\geqslant r)\\
=&\frac{n!}{m_1!\cdots m_{r+1}!}[P(X=k_1]^{m_1}\cdots[P(X=k_r]^{m_r}[P(X>k_r]^{m_{r+1}}\\
=&\frac{n!}{m_1!\cdots m_{r+1}!}(pq^{k_1-1})^{m_1}\cdots(pq^{k_r-1})^{m_r}(\sum_{k=k_r+1}^{\infty}pq^{k-1})^{m_{r+1}}\\
=&\frac{n!}{m_1!\cdots m_{r+1}!}p^{\sum_{i=1}^{r}m_i}q^{\sum_{i=1}^{r}m_ik_i+m_{r+1}k_r-\sum_{i=1}^{r}m_i},
\end{aligned}\tag{6}$$

where $m_1+\cdots+m_{r+1}=n$ and $1\leqslant k_1<\cdots<k_r$.

From Equation (6), one has

$$\begin{aligned}
&P(X_{(r)}=k_r,Y_r=m_r,N\geqslant r|X_{(1)}=k_1,Y_1=m_1,\cdots,X_{(r-1)}=k_{r-1},Y_{r-1}=m_{r-1},N\geqslant r-1)\\
=&\frac{P(X_{(1)}=k_1,Y_1=m_1,\cdots,X_{(r)}=k_r,Y_r=m_r,N\geqslant r)}{P(X_{(1)}=k_1,Y_1=m_1,\cdots,X_{(r-1)}=k_{r-1},Y_{r-1}=m_{r-1},N\geqslant r-1)}\\
=&\binom{n-m_1-\cdots-m_{r-1}}{m_r}(\frac{p}{q})^{m_r}(q^{n-m_1-\cdots-m_{r-1}})^{k_r-k_{r-1}},
\end{aligned}$$

therefore,

$$\begin{aligned}
&P(X_{(r)}=k_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1},Y_{r-1}=m_{r-1},N\geqslant r-1)\\
=&\sum_{m_r=1}^{n-m_1-\cdots-m_{r-1}}\binom{n-m_1-\cdots-m_{r-1}}{m_r}(q^{n-m_1-\cdots-m_{r-1}})^{k_r-k_{r-1}}\\
=&(q^{n-m_1-\cdots-m_{r-1}})^{k_r-k_{r-1}-1}[(\frac{p}{q}+1)^{n-m_1-\cdots-m_{r-1}}-1]\\
=&(1-q^{n-m_1-\cdots-m_{r-1}})(q^{n-m_1-\cdots-m_{r-1}})^{k_r-k_{r-1}-1},
\end{aligned}$$

hence,

$$\begin{aligned}
&P(X_{(r)}-X_{(r-1)}=k,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1},Y_{r-1}=m_{r-1},N\geqslant r-1)\\
=&(1-q^{n-m_1-\cdots-m_{r-1}})(q^{n-m_1-\cdots-m_{r-1}})^{k-1}.
\end{aligned}\tag{7}$$

Obviously, the right of Equation (7) has nothing to do with k_{r-1}, hence we have

$$\begin{aligned}
&P(X_{(r)}-X_{(r-1)}=k,N\geqslant r|Y_1=m_1,\cdots,Y_{r-1}=m_{r-1},N\geqslant r-1)\\
=&(1-q^{n-m_1-\cdots-m_{r-1}})(q^{n-m_1-\cdots-m_{r-1}})^{k-1}.
\end{aligned}$$

That is to say: Conditional $Y_1=m_1,\cdots,Y_{r-1}=m_{r-1}$, $D_r\sim Geom(1-q^{n-m_1-\cdots-m_{r-1}})$.

(iii) Firstly, it can be obtained that

$$\begin{aligned}
&P(Y_r=m_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1},Y_{r-1}=m_{r-1},N\geqslant r-1)\\
=&\sum_{k_r=k_{r-1}+1}^{\infty}\binom{n-m_1-\cdots-m_{r-1}}{m_r}(q^{n-m_1-\cdots-m_{r-1}})^{k_r-k_{r-1}}\\
=&\binom{n-m_1-\cdots-m_{r-1}}{m_r}(\frac{p}{q})^{m_r}\frac{q^{n-m_1-\cdots-m_{r-1}}}{1-q^{n-m_1-\cdots-m_{r-1}}}\\
=&P(Y_r=m_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-1}=m_{r-1},N\geqslant r-1),
\end{aligned}$$

therefore, conditional $Y_1=m_1,\cdots,Y_{r-1}=m_{r-1}$, the frequency function of Y_r is

$$P(Y_r=m_r|Y_1=m_1,\cdots,Y_{r-1}=m_{r-1})=\binom{n-m_1-\cdots-m_{r-1}}{m_r}(\frac{p}{q})^{m_r}\frac{q^{n-m_1-\cdots-m_{r-1}}}{1-q^{n-m_1-\cdots-m_{r-1}}}.$$

(iv) Since

$$\begin{aligned} & P(X_{(r)}=k_r,Y_r=m_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1}) \\ = & P(X_{(r)}=k_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1}) \\ & \times P(Y_r=m_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-2}=m_{r-2},X_{(r-1)}=k_{r-1}), \end{aligned}$$

hence, conditional $Y_1=m_1,\cdots,X_{r-1}=k_{r-1},Y_{r-1}=m_{r-1}$, variables $X_{(r)}$ and Y_r are independent.

(v) Define $A=\frac{n!}{m_1!\cdots m_{r+1}!}(pq^{-1})^{\sum_{i=1}^r m_i}$, it can be obtained that

$$\begin{aligned} P(Y_1=m_1,\cdots,Y_r=m_r,N\geqslant r) &= \sum_{k_1=1}^{\infty}\sum_{k_2=k_1+1}^{\infty}\cdots\sum_{k_r=k_{r-1}+1}^{\infty} Aq^{\sum_{i=1}^r m_ik_i+m_{r+1}k_r} \\ &= A\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+1}^{\infty}\cdots\sum_{k_r=k_{r-1}+1}^{\infty} q^{\sum_{i=1}^{r-1} m_ik_i+(n-m_1-\cdots-m_{r-1})k_{r-1}} \\ &= A\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+1}^{\infty}\cdots\sum_{k_{r-1}=k_{r-2}+1}^{\infty} q^{\sum_{i=1}^{r-2} m_ik_i}\frac{q^{(n-m_1-\cdots-m_{r-2})k_{r-1}}}{1-q^{n-m_1-\cdots m_{r-1}}} \\ &= \frac{Aq^{n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})}}{(1-q^n)(1-q^{n-m_1})\cdots(1-q^{n-m_1-\cdots-m_{r-1}})}\sum_{k_1=1}^{\infty}q^{nk_i} \\ &= \frac{n!}{m_1!\cdots m_{r+1}!}\frac{p^{\sum_{i=1}^r m_i}q^{n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})-\sum_{i=1}^r m_i}}{(1-q^n)(1-q^{n-m_1})\cdots(1-q^{n-m_1-\cdots-m_{r-1}})}, \end{aligned}$$

hence,

$$P(Y_1=m_1,\cdots,Y_r=m_r,N\geqslant r)=\frac{n!}{m_1!\cdots m_{r+1}!}\frac{p^{\sum_{i=1}^r m_i}q^{n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})-\sum_{i=1}^r m_i}}{(1-q^n)(1-q^{n-m_1})\cdots(1-q^{n-m_1-\cdots-m_{r-1}})},$$

where $m_1+\cdots+m_{r+1}=n$.

The other solution is as follows.

It is easy to check that

$$P(X_1=\cdots=X_l<\min\{X_{l+1},\cdots,X_{l+m}\})=\sum_{k=1}^{\infty}(pq^{k-1})^l(q^k)^m=(\frac{p}{q})^l\frac{q^{l+m}}{1-q^{l+m}},$$

hence,

$$\begin{aligned} & P(Y_1=m_1,\cdots,Y_r=m_r,n\geqslant r) \\ = & P(Y_1=m_1)P(Y_2=m_2,N\geqslant 2|Y_1=m_1)\cdots P(Y_r=m_r,N\geqslant r|Y_1=m_1,\cdots,Y_{r-1}=m_{r-1},N\geqslant r-1) \\ = & \binom{n}{m_1}(\frac{p}{q})^{m_1}\frac{q^n}{1-q^n}\binom{n-m_1}{m_2}(\frac{p}{q})^{m_2}\frac{q^{n-m_1}}{1-q^{n-m_1}}\cdots\binom{n-m_1-\cdots-m_{r-1}}{m_r}(\frac{p}{q})^{m_r}\frac{q^{n-m_1-\cdots-m_{r-1}}}{1-q^{n-m_1-\cdots-m_{r-1}}} \\ = & \frac{n!}{m_1!\cdots m_{r+1}!}\frac{p^{\sum_{i=1}^r m_i}q^{n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})-\sum_{i=1}^r m_i}}{(1-q^n)(1-q^{n-m_1})\cdots(1-q^{n-m_1-\cdots-m_{r-1}})}. \end{aligned}$$

Let $m_1=m_2=\cdots=m_r=1$ above, it follows that

$$P(X_i\neq X_j \text{ for } i\neq j,i,j=1,2,\cdots,n)=P(Y_1=1,\cdots,Y_n=1)=n!p^nq^{\frac{n(n-1)}{2}}\prod_{i=1}^n\frac{1}{1-q^i}.$$

(vi) From Equations (4) and (6), one has

$$\begin{aligned} & P(X_{(1)}=k_1,\cdots,X_{(r)}=k_r|Y_1=m_1,\cdots,Y_r=m_r,N\geqslant r) \\ = & \frac{P(X_{(1)}=k_1,\cdots,X_{(r)}=k_r,Y_1=m_1,\cdots,Y_r=m_r,N\geqslant r)}{P(Y_1=m_1,\cdots,Y_r=m_r,N\geqslant r)} \\ = & (1-q^n)(1-q^{n-m_1})\cdots(1-q^{n-m_1-\cdots m_{r-1}})q^{\sum_{i=1}^r m_ik_i+(n-m_1-\cdots-m_r)k_r-[n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})]}, \end{aligned}$$

where $m_1 + \cdots + m_r \leqslant n$ and $1 \leqslant k_1 < \cdots < k_r$.

Therefore,

$$\begin{aligned}
& P(D_{(1)} = y_1, \cdots, D_{(r)} = y_r | Y_1 = m_1, \cdots, Y_r = m_r, N \geqslant r) \\
= & \frac{P(X_{(1)} = y_1, X_{(2)} = y_1 + y_2 \cdots, X_{(r)} = y_1 + \cdots + y_r, Y_1 = m_1, \cdots, Y_r = m_r, N \geqslant r)}{P(Y_1 = m_1, \cdots, Y_r = m_r, N \geqslant r)} \\
= & (1-q^n)(1-q^{n-m_1}) \cdots (1-q^{n-m_1-\cdots m_{r-1}}) \\
& \times q^{[m_1 y_1 + m_2(y_1+y_2) + \cdots + m_r(y_1+\cdots+y_r) + (n-m_1-\cdots-m_r)(y_1+\cdots+y_r)] - [n+(n-m_1)+\cdots+\cdots+(n-m_1-\cdots-m_{r-1})]} \\
= & (1-q^n)(1-q^{n-m_1}) \cdots (1-q^{n-m_1-\cdots m_{r-1}}) \\
& \times q^{[ny_1 + (n-m_1)y_2 + \cdots + (n-m_1-\cdots-m_{r-1})y_r] - [n+(n-m_1)+\cdots+(n-m_1-\cdots-m_{r-1})]} \\
= & [(1-q^n)(q^n)^{y_1-1}][(1-q^{n-m_1})(q^{n-m_1})^{y_2-1}] \cdots [(1-q^{n-m_1-\cdots m_{r-1}})(q^{n-m_1-\cdots m_{r-1}})^{y_r-1}],
\end{aligned}$$

where $m_1 + m_2 + \cdots + m_r \leqslant n$.

Hence, conditional $Y_1 = m_1, \cdots, Y_r = m_r$, variables $D_1, \cdots, D_r$ are independent and

$$D_i \sim Geo(1 - q^{n-m_1-\cdots-m_{i-1}}),\ i = 1, 2 \cdots, r.$$

Let $m_1 = m_2 = \cdots = m_r = 1$ above, it follows that conditional $X_i \neq X_j (i \neq j, i, j = 1, 2, \cdots, n)$

$$D_i \sim Geo(1 - q^{n-i+1}),\ i = 1, 2 \cdots, n.$$

The proof is complete.

Proposition 2: Suppose that $Y_1, Y_2, \cdots, Y_n$ are independent with $Y_i \sim Geo(1 - q^i), i = 1, 2, \cdots, n$, then conditional $X_i \neq X_j (i \neq j)$,

$$X_{(n-k+1)} \text{ and } \sum_{i=k}^{n} Y_i \text{ have an identical distribution.}$$

Proof: Conditional $X_i \neq X_j (i \neq j)$, variables Y_i and $D_{(n+1-i)}$ have an identical distribution, hence

$$\sum_{i=k}^{n} Y_i \text{ and } \sum_{i=k}^{n} D_{(n+1-i)} \text{ have an identical distribution.}$$

Moreover,

$$\begin{aligned}
\sum_{i=k}^{n} D_{(n+1-i)} &= \sum_{i=k}^{n} (X_{(n+1-i)} - X_{(n-i)}) \\
&= X_{(n+1-k)}.
\end{aligned}$$

From the above, it follows that

$$X_{(n-k+1)} \text{ and } \sum_{i=k}^{n} Y_i \text{ have an identical distribution.}$$

The proof is complete. ■

Corollary 1: Suppose that $Z_1, Z_2, \cdots, Z_k$ are i.d.d. geometric variables with $k < n$, then $X_{(n-l)} - X_{(n-k)}$ given $X_i \neq X_j (i \neq j)$ and $Z_{(k-l)}$ given $Z_i \neq Z_j (i \neq j)$ are identically distributed.

Proof: it is easy to check that

$$X_{(n-l)} = \sum_{i=l+1}^{n} Y_i \quad \text{and} \quad X_{(n-k)} = \sum_{i=k+1}^{n} Y_i \, ,$$

hence,

$$X_{(n-l)} - X_{(n-k)} = \sum_{i=l+1}^{n} Y_i.$$

Since

$$Z_{(k-l)} \text{ and } \sum_{i=l+1}^{k} Y_i \text{ have an identical distribution,}$$

therefore,

$$X_{(n-l)} - X_{(n-k)} \text{ and } Z_{(k-l)} \text{ are identically distributed.}$$

The proof is complete. ■

Corollary 2: X_n given $X_i \neq X_j (i \neq j)$ and $\sum_{i=l+1}^{k} Y_i$ have an identical distribution.

Corollary 3: conditional $X_i \neq X_j (i \neq j)$, variables $X_{(n)} - X_{(1)}$ and max $(X_1, \cdots, X_n)$ have an identical distribution.

III. CONCLUSION

The current work concerns the order statistics of geometric distribution. The joint frequency function and conditional joint frequency function of the order statistics has been obtained by applying properties of probability and combinatorial arguments. Severl propositions of order statistics are very fresh, interesting and attractive. Results indicate that certain propositions are different from and also similar to corresponding propositions of exponential distribution. According to the theoretical conclusions of this paper, further topics will include the parameter estimation on the basis of observation data.

IV. CONFLICT OF INTEREST

The author declares that there is no conflict of interest regarding the publication of this paper.

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