

SEPARATION AXIOMS AND CONNECTEDNESS FOR $\beta\omega$ -OPEN SETS

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Abstract:-

Our propose in this paper is to introduce the new classes for separation axioms in topo logical spaces by using $\beta\omega$ -open sets and $G_{\beta\omega}$ -open sets, called $\beta\omega$ -separation axioms and $G_{\beta\omega}$ -separation axioms. Furthermore, we introduce the stronger form of connected spaces.

Keywords:-Open set; Generalized closed set; Connectedness.

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1 INTRODUCTION

In 1970 Levine, [7], introduced the notion of a generalized closed set. A subset A of a space X is called a generalized closed set (simply g -closed set) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of a generalized closed set (simply g -open set) is called a generalized open set. In 1982 Hdeib [5] introduced the notion of a ω -open sets. A subset A of a space X is called ω -open set if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a ω -open set is called a ω -closed set. In 1983 the authors [1] introduced the weak form for an open set which is called a β -open set. A subset A of a space X is called a β -open set if $A \subseteq Cl(Int(Cl(A)))$. The complement of a β -open set is called a β -closed set. In 2005 Al-Zoubi [2] introduced the generalization property of ω -open sets. A subset A of a space X is called generalized ω -closed set if $Cl_\omega(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of generalized ω -closed set is called generalized ω -open set, where $Cl_\omega(A)$ is the ω -closure set of A . In 2009 Noiri and Noorani [8] introduced the notion of $\beta\omega$ -open set as weak form for a ω -open sets and a β -open sets. A subset A of a space X is called a $\beta\omega$ -open set if $A \subseteq Cl(Int_\omega(Cl(A)))$. The complement of a $\beta\omega$ -open set is called a $\beta\omega$ -closed set, where $Int_\omega(A)$ is the ω -interior set of A . In 2019 [9] we introduced the notion of $G_{\beta\omega}$ -closed set as weak form for a $\beta\omega$ -closed sets and a β -open sets. A subset A of a topological space (X, τ) is called generalized $\beta\omega$ -closed (simply $G_{\beta\omega}$ -closed) set if $Cl_{\beta\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ) . The complement of $G_{\beta\omega}$ -closed set is called generalized $\beta\omega$ -open (simply $G_{\beta\omega}$ -open) set, where $Cl_{\beta\omega}(A)$ is the $\beta\omega$ -closure set of A which defined as the intersection of all $\beta\omega$ -closed subsets of X containing A . Similar, the $\beta\omega$ -interior set of A is defined as the union of all $\beta\omega$ -open subsets of X contained in A and is denoted by $Int_{\beta\omega}(A)$.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we introduce the new classes for separation axioms in topological spaces, called $\beta\omega$ -separation axioms. Furthermore, the relationship with the other known axioms will be studied. In

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Section 4 we introduce also the new classes for separation axioms in topological spaces, called $G_{\beta\omega}$ -separation axioms. Furthermore, the relationship with the other known axioms will be also studied. In Section 5 we introduce the stronger form of connected spaces.

2 Preliminaries

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A , respectively.

A subset of topological space is called a *clopen* set if it is both open and closed set. A topological space (X, τ) is called *0-dimensional space*, [6] if it has a base consisting clopen sets.

Definition 2.1. [6] A topological space (X, τ) is called a *disconnected space* if it is the union of two nonempty subsets A and B such that $Cl(A) \cap B = \emptyset$ and $A \cap Cl(B) = \emptyset$.

Theorem 2.2. [6] A topological space (X, τ) is a disconnected space if and only if it is the union of two disjoint nonempty open subsets.

Theorem 2.3. [6] For a topological space (X, τ) and $A, B \subseteq X$, if B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

Theorem 2.4. [7] Every closed set is a g -closed set.

Definition 2.5. [7] A topological space (X, τ) is called a $T_{1/2}$ -space if every g -closed set is closed set.

Theorem 2.6. [4] A topological space (X, τ) is $T_{1/2}$ -space if and only if every singleton set is open or closed set.

Definition 2.7. [6] A topological space (X, τ) is called:

1. T_0 -space if for two points $x \neq y \in X$ in X , there is open set G in X such that $x \in G$ and $y \notin G$.
2. T_1 -space if for two points $x \neq y \in X$ in X , there are two open sets G and U in X such that $x \in G, y \notin G, y \in U$ and $x \notin U$.
3. T_2 -space or Hausdorff space if for two points $x \neq y \in X$ in X , there are two open sets G and U in X such that $x \in G, y \in U$ and $U \cap G = \emptyset$.
4. *regular space* if for each closed set F in X and each $x \notin F$, there are two open sets G and U in X such that $F \subseteq G, x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called T_3 -space if it is regular space and T_1 -space.
5. *Normal space* if for each two disjoint closed sets F and M in X , there are two open sets G and U in X such that $F \subseteq G, M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called T_4 -space if it is normal space and T_1 -space.

Theorem 2.8. [6] A topological space (X, τ) is T_1 -space if and only if every singleton set is closed set.

Theorem 2.9. [6] A topological space (X, τ) is regular space if and only if for each $x \in X$ and for each open set N in X containing x , there is an open set M in X containing x such that $Cl(M) \subseteq N$.

Theorem 2.10. [5] Every open set is ω -open set.

Theorem 2.11. [5] For a topological space (X, τ) , the collection of all ω -open sets with a set X forms a topological space.

Theorem 2.12. [8] The union of arbitrary of $\beta\omega$ -open sets is $\beta\omega$ -open set.

Theorem 2.13. [8] Every ω -open set is $\beta\omega$ -open set.

Definition 2.14. [6] A function $f: (X, \tau) \rightarrow (Y, \rho)$ of a space (X, τ) into a space (Y, ρ) is called *continuous function* if $f^{-1}(U)$ is an open set in X for every open set U in Y .

Definition 2.15. A function $f: (X, \tau) \rightarrow (Y, \rho)$ of a space (X, τ) into a space (Y, ρ) is called:

1. *open function* [6] if $f(U)$ is open set in Y for every open set U in X .
2. *closed function* [6] if $f(U)$ is closed set in Y for every closed set U in X .
3. *$\beta\omega$ -continuous function* [9] if $f^{-1}(U)$ is a $\beta\omega$ -open set in X for every open set U in Y .

Theorem 2.16. [9] Every $\beta\omega$ -open set is $G_{\beta\omega}$ -open set.

Theorem 2.17. [9] Let (X, τ) be a topological space. If (X, τ) is a $T_{1/2}$ -space then every $G_{\beta\omega}$ -closed set in X is $\beta\omega$ -closed set in X .

3 $\beta\omega$ -Separation axioms

Definition 3.1. A topological space (X, τ) is called:

1. $\beta\omega^2$ -space if for two points $x \neq y \in X$ in X , there are two $\beta\omega$ -open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
2. $\beta\omega$ -regular space if for each closed set F in X and each $x \notin F$, there are two $\beta\omega$ -open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $\beta\omega^3$ -space if it is $\beta\omega$ -regular space and T_1 -space.
3. $\beta\omega$ -normal space if for each two disjoint closed sets F and M in X , there are two $\beta\omega$ -open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $\beta\omega^4$ -space if it is $\beta\omega$ -normal space and T_1 -space.

The proof of the following theorem, Theorem (3.3) and Theorem (3.4) follow from the fact that open sets are $\beta\omega$ -open sets.

Theorem 3.2. Every T_2 -space is a $\beta\omega^2$ -space.

Theorem 3.3. Every regular space is a $\beta\omega$ -regular space.

Theorem 3.4. Every normal space is a $\beta\omega$ -normal space.

The converse of the Theorems (3.2), (3.3) and (3.4) need not be true.

Example 3.5. Let $X = \{1, 2, 3\}$. The indiscrete topological space (X, T_1) , where $T_1 = \{\emptyset, X\}$, is $\beta\omega^2$ -space, $\beta\omega$ -regular space and $\beta\omega$ -normal space, since all subsets of countable topological space are $\beta\omega$ -open sets, but it is not T_2 -space, regular space or normal space.

Theorem 3.6. Every $\beta\omega^3$ -space is a $\beta\omega^2$ -space.

Proof. Let (X, τ) be a $\beta\omega^3$ -space and $x \neq y \in X$ be any points in X . Since X is a T_1 -space then by Theorem (2.8), $\{x\}$ is a closed set in X and $y \notin \{x\}$. Since X is a $\beta\omega$ -regular space then there are two $\beta\omega$ -open sets G and U in X such that $x \in \{x\} \subseteq G$, $y \in U$ and $U \cap G = \emptyset$. Hence X is a $\beta\omega^2$ -space. $\square$

Theorem 3.7. Every $\beta\omega^4$ -space is a $\beta\omega^3$ -space.

Proof. Let (X, τ) be a $\beta\omega^4$ -space. Let F be any closed set in X and $x \notin F$ be any points in X . Since X is a T_1 -space then by Theorem (2.8), $\{x\}$ is a closed set in X and $F \cap \{x\} = \emptyset$. Since X is a $\beta\omega$ -normal space then there are two $\beta\omega$ -open sets G and U in X such that $x \in \{x\} \subseteq G$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence X is a $\beta\omega^3$ -space. $\square$

We have the following relation.

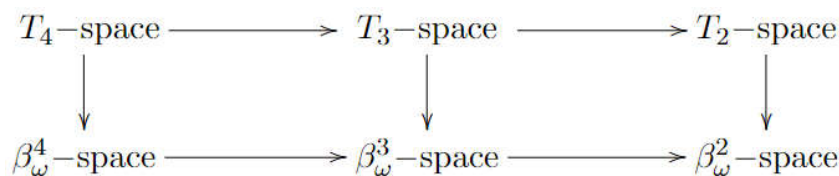


Figure 1:

Theorem 3.8. A topological space (X, τ) is a $\beta\omega^2$ -space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $\beta\omega$ -open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.

Proof. Suppose that (X, τ) is $\beta\omega^2$ -space. Let $x \in X$ be any point in X and $y \neq x$ be any point in X . Then there are two $\beta\omega$ -open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Take $M = G$ is a $\beta\omega$ -open set in X containing x and so $y \notin M \subseteq Cl_{\beta\omega}(M)$.

Conversely, Let $x \neq y \in X$ be any points in X . By the hypothesis, there is a $\beta\omega$ -open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$. Then $X - Cl_{\beta\omega}(M)$ is a $\beta\omega$ -open set M in X containing y and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$. Then (X, τ) is $\beta\omega^2$ -space.

Theorem 3.9. A topological space (X, τ) is a $\beta\omega$ -regular space if and only if for each $x \in X$ and for each open set N in X containing x , there is a $\beta\omega$ -open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.

Proof. Suppose that (X, τ) is $\beta\omega$ -regular space. Let $x \in X$ be any point in X and N be any open set in X containing x . Then $X - N$ is a closed set in X and $x \notin X - N$. Since (X, τ) is $\beta\omega$ -regular space then there are two $\beta\omega$ -open sets G and U in X such that $X - N \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Take $M = U$ is a $\beta\omega$ -open set in X containing x . Then $M = U \subseteq X - G$, this implies,

$$Cl_{\beta\omega}(M) \subseteq Cl_{\beta\omega}(X - G) = X - G \subseteq N.$$

Conversely, Let F be any closed set in X and $x \notin F$. Then $x \in X - F$ and $X - F$ is an open set in X containing x . By the hypothesis, there is a $\beta\omega$ -open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(M)$ and $X - Cl_{\beta\omega}(M)$ is a $\beta\omega$ -open set in X . Since M is a $\beta\omega$ -open set in X containing x and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$, then (X, τ) is $\beta\omega$ -regular space. $\square$

Theorem 3.10. A topological space (X, τ) is $\beta\omega$ -normal space if and only if for each closed set F in X and for each open set G in X containing F , there is a $\beta\omega$ -open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.

Proof. Suppose that (X, τ) is $\beta\omega$ -normal space. Let F be any closed set in X and G be any open set in X containing F . Then $X - G$ is a closed set in X and $F \cap (X - G) = \emptyset$. Since (X, τ) is $\beta\omega$ -normal space then there are two $\beta\omega$ -open sets H and U in X such that $X - G \subseteq U$, $F \subseteq H$ and $U \cap H = \emptyset$. Take $V = H$ is a $\beta\omega$ -open set in X containing F . Then $V = H \subseteq X - U$, this implies,

$$Cl_{\beta\omega}(V) \subseteq Cl_{\beta\omega}(X - U) = X - U \subseteq G.$$

Conversely, Let F and M be any two closed sets in X such that $F \cap M = \emptyset$. Then $M \subseteq X - F$ and $X - F$ is an open set in X containing closed set M . By the hypothesis, there is a $\beta\omega$ -open set V in X containing M such that $Cl_{\beta\omega}(V) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(V)$ and $X - Cl_{\beta\omega}(V)$ is a $\beta\omega$ -open set in X . Since V is a $\beta\omega$ -open set in X containing x and $V \cap [X - Cl_{\beta\omega}(V)] = \emptyset$, then (X, τ) is $\beta\omega$ -normal space. $\square$

Theorem 3.11. If a function $f: (X, \tau) \rightarrow (Y, \rho)$ is $\beta\omega$ -continuous injection and Y is a T_2 -space then X is a $\beta\omega^2$ -space.

Proof. Let Y be a T_2 -space and $x \neq y \in X$ be any points in X . Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $\beta\omega$ -continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega$ -open sets in X . Hence X is a $\beta\omega^2$ -space. $\square$

A subset of topological space is called a $\beta\omega$ -clopen set if it is both $\beta\omega$ -open and $\beta\omega$ -closed set. sets.

Definition 3.12. A function $f: (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is called *slightly $\beta\omega$ -continuous function* if $f^{-1}(U)$ is a $\beta\omega$ -clopen set in X for every clopen set U in Y .

Theorem 3.13. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be a slightly $\beta\omega$ -continuous injection function and Y be 0-dimensional. If Y is a T_2 -space then X is a $\beta\omega^2$ -space.

Proof. Let Y be a T_2 -space and $x \neq y \in X$ be any points in X . Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Since Y is 0-dimensional space there are two clopen sets G_1 and U_1 in Y such that

$$f(x) \in G_1 \subseteq G \quad \text{and} \quad f(y) \in U_1 \subseteq U.$$

Then $x \in f^{-1}(G_1) \subseteq f^{-1}(G)$ and $y \in f^{-1}(U_1) \subseteq f^{-1}(U)$.

and $f^{-1}(G_1) \cap f^{-1}(U_1) \subseteq f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset$.

Since G_1 and U_1 are clopen sets in Y and f is a slightly $\beta\omega$ -continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega$ -open sets in X . Hence X is a $\beta\omega^2$ -space. $\square$

Theorem 3.14. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be $\beta\omega$ -continuous injection function. If f is an open (or closed) function and Y is a regular space then X is a $\beta\omega$ -regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x . Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y . Since Y is a regular space then by Theorem(2.9), there is an open set M in Y containing $f(x)$ such that $Cl(M) \subseteq f(U)$. Since f is a $\beta\omega$ -continuous then $V = f^{-1}(M)$ is a $\beta\omega$ -open set in X containing x . Since f is injection then

$$f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Then

$$Cl_{\beta\omega}(V) = Cl_{\beta\omega}[f^{-1}(M)] \subseteq f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Hence by Theorem (3.9), X is a $\beta\omega$ -regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y . Since Y is a regular space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(x) \in U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $x \notin f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $\beta\omega$ -continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $\beta\omega$ -open in X . Hence X is a $\beta\omega$ -regular space.

Theorem 3.15. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be slightly $\beta\omega$ -continuous injection and Y is 0-dimensional space. If f is an open (or closed) function then X is a $\beta\omega$ -regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x . Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y . Since Y is a 0-dimensional space then there is a clopen set V in Y such that $f(x) \in V \subseteq f(U)$. Since f is injection then $x \in f^{-1}(V) \subseteq U$. Since f is a $\beta\omega$ -continuous then $f^{-1}(V)$ is a $\beta\omega$ -clopen set in X containing x . Hence

$$Cl_{\beta\omega}(f^{-1}(V)) = f^{-1}(V) \subseteq U.$$

Hence by Theorem (3.9), X is a $\beta\omega$ -regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y . Then $f(x) \in Y - f(F)$ and $Y - f(F)$ is an open set in Y . Since Y is a 0-dimensional space then there is a clopen set V in Y such that $f(x) \in V \subseteq Y - f(F)$. Since f is injection then

$$x \in f^{-1}(V) \subseteq f^{-1}[Y - f(F)] \subseteq X - F.$$

Since f is a slightly $\beta\omega$ -continuous then $f^{-1}(V)$ is a $\beta\omega$ -clopen set in X containing x and $X - f^{-1}(V)$ is a $\beta\omega$ -clopen set in X such that $F \subseteq X - f^{-1}(V)$. Hence X is a $\beta\omega$ -regular space. $\square$

Theorem 3.16. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be $\beta\omega$ -continuous injection function. If f is closed function and Y is a normal space then X is a $\beta\omega$ -normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $\beta\omega$ -continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $\beta\omega$ -open in X . Hence X is a $\beta\omega$ -normal space. $\square$

Theorem 3.17. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be slightly $\beta\omega$ -continuous injection and Y is 0 dimensional space. If f is a closed function and Y is a normal space then X is a $\beta\omega$ -normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since Y is a 0-dimensional space then for every $g \in f(F)$ and $u \in f(H)$ there are clopen sets U_u and G_g in Y such that

$$u \in U_u \subseteq U \quad \text{and} \quad g \in G_g \subseteq G.$$

Then $f(H) \subseteq \bigcup \{U_u : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\} \subseteq U$
and $f(F) \subseteq \bigcup \{G_g : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\} \subseteq G$.

This implies,

$$H \subseteq \bigcup \{f^{-1}(U_u) : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\} \subseteq f^{-1}(U)$$

and

$$F \subseteq \bigcup \{f^{-1}(G_g) : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\} \subseteq f^{-1}(G).$$

Since f is a slightly $\beta\omega$ -continuous then $f^{-1}(U_u)$ and $f^{-1}(G_g)$ are $\beta\omega$ -open in X for all $g \in f(F)$ and $u \in f(H)$. So that

$$M = \bigcup \{f^{-1}(U_u) : u \in f(H)\} \quad \text{and} \quad N = \bigcup \{f^{-1}(G_g) : g \in f(F)\}$$

are $\beta\omega$ -open in X and

$$M \cap N \subseteq f^{-1}(U) \cap f^{-1}(G) \subseteq f^{-1}(U \cap G) = f^{-1}(\emptyset) = \emptyset.$$

Hence X is a $\beta\omega$ -normal space. $\square$

4 $G_{\beta\omega}$ -Separation axioms

Definition 4.1. A topological space (X, τ) is called:

1. $G^2_{\beta\omega}$ -space if for two points $x \neq y \in X$ in X , there are two $G_{\beta\omega}$ -open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
2. $G_{\beta\omega}$ -regular space if for each closet set F in X and each $x \notin F$, there are two $G_{\beta\omega}$ -open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $G^3_{\beta\omega}$ -space if it is $G_{\beta\omega}$ -regular space and T_1 -space.
3. $G_{\beta\omega}$ -normal space if for each two disjoint closet sets F and M in X , there are two $G_{\beta\omega}$ -open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $G^4_{\beta\omega}$ -space if it is $G_{\beta\omega}$ -normal space and T_1 -space.

It is clear that every β_ω^2 -space is a $G_{\beta\omega}^2$ -space, every $\beta\omega$ -regular space is a $G_{\beta\omega}$ -regular space and every $\beta\omega$ -normal space is a $G_{\beta\omega}$ -normal space.

Theorem 4.2. Every $G_{\beta\omega}^3$ -space is a $G_{\beta\omega}^2$ -space.

Proof. Similar for Theorem (3.6). □

Theorem 4.3. Every $G_{\beta\omega}^4$ -space is a $G_{\beta\omega}^3$ -space.

Proof. Similar for Theorem (3.7). □

Theorem 4.4. Let (X, τ) be a $T_{1/2}$ -space. If X is a $G_{\beta\omega}^2$ -space then X is a β_ω^2 -space.

Proof. For two points $x \neq y \in X$ in X , since X is a $G_{\beta\omega}^2$ -space, there are two $G_{\beta\omega}$ -open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$ -space, then by Theorem (2.17), G and U are $\beta\omega$ -open sets in X . Hence X is a β_ω^2 -space. □

Theorem 4.5. Let (X, τ) be a $T_{1/2}$ -space. If X is a $G_{\beta\omega}$ -regular space then X is a $\beta\omega$ -regular space

Proof. For each closet set F in X and each $x \in F$, since X is a $G_{\beta\omega}$ -regular space, there are two $G_{\beta\omega}$ -open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$ -space, then by Theorem (2.17), G and U are $\beta\omega$ -open sets in X . Hence X is a $\beta\omega$ -regular space. □

Corollary 4.6. Every $G_{\beta\omega}^3$ -space is a β_ω^3 -space.

Proof. Use above theorem, since every T_1 -space is $T_{1/2}$ -space. □

Theorem 4.7. Let (X, τ) be a $T_{1/2}$ -space. If X is a $G_{\beta\omega}$ -normal space then X is a $\beta\omega$ -normal space

Proof. For each two disjoint closet sets F and M in X , since X is a $G_{\beta\omega}$ -normal space, there are two $G_{\beta\omega}$ -open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$ -space then by Theorem (2.17), G and U are $\beta\omega$ -open sets in X . Hence X is a $\beta\omega$ -normal space. □

Corollary 4.8. Every $G_{\beta\omega}^4$ -space is a β_ω^4 -space.

Proof. Use above theorem, since every T_1 -space is $T_{1/2}$ -space. □

We have the following relation.

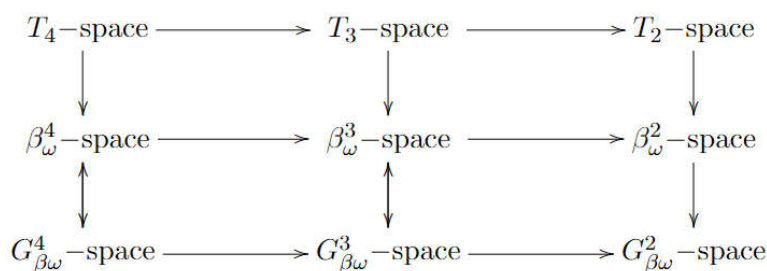


Figure 2:

Theorem 4.9. A topological space (X, τ) is $G_{\beta\omega}^2$ -space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $G_{\beta\omega}$ -open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.

Proof. Similar for Theorem (3.8). □

Theorem 4.10. A topological space (X, τ) is $G_{\beta\omega}$ -regular space if and only if for each $x \in X$ and for each open set N in X containing x , there is a $G_{\beta\omega}$ -open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.

Proof. Similar for Theorem (3.9). □

Theorem 4.11. A topological space (X, τ) is $G_{\beta\omega}$ -normal space if and only if for each closed set F in X and for each open set G in X containing F , there is a $G_{\beta\omega}$ -open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.

Proof. Similar for Theorem (3.10). □

Theorem 4.12. If a function $f: (X, \tau) \rightarrow (Y, \rho)$ is $G_{\beta\omega}$ -continuous injection and Y is a T_2 -space then X is a $G_{\beta\omega}^2$ -space.

Proof. Let Y be a T_2 -space and $x \neq y \in X$ be any points in X . Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $G_{\beta\omega}$ -continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $G_{\beta\omega}$ -open sets in X . Hence X is a $G_{\beta\omega}^2$ -space. □

Theorem 4.13. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be $G_{\beta\omega}$ -continuous injection function. If f is an open (or closed) function and Y is a regular space then X is a $G_{\beta\omega}$ -regular space.

Proof. 1. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x . Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y . Since Y is a regular space then by Theorem(2.9), there is an open set M in Y containing $f(x)$ such that $Cl(M) \subseteq f(U)$. Since f is a $G_{\beta\omega}$ -continuous then $V = f^{-1}(M)$ is a $G_{\beta\omega}$ -open set in X containing x . Since f is injection then

$$f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Hence

$$Cl_{\beta\omega}(V) = Cl_{\beta\omega}[f^{-1}(M)] \subseteq f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Then by Theorem (4.10), X is a $G_{\beta\omega}$ -regular space.

2. Secondly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y . Since Y is a regular space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(x) \in U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $x \notin f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $G_{\beta\omega}$ -continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $G_{\beta\omega}$ -open in X . Hence X is a $G_{\beta\omega}$ -regular space.

Theorem 4.14. Let $f: (X, \tau) \rightarrow (Y, \rho)$ be $G_{\beta\omega}$ -continuous injection function. If f is closed function and Y is a normal space then X is a $G_{\beta\omega}$ -normal space.

Proof. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $G_{\beta\omega}$ -continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $G_{\beta\omega}$ -open in X . Hence X is a $G_{\beta\omega}$ -normal space. $\square$

5 $\beta\omega$ -Connectedness property

Definition 5.1. Let (X, τ) be a topological space and A, B be two nonempty subsets of X . The sets A and B are called a $\beta\omega$ -separated sets if $Cl_{\beta\omega}(A) \cap B = \emptyset$ and $A \cap Cl_{\beta\omega}(B) = \emptyset$.

Remark 5.2. Let (X, τ) be a topological space. Then

1. Any $\beta\omega$ -separated sets are disjoint sets, since $A \cap B \subseteq A \cap Cl_{\beta\omega}(B) = \emptyset$.
2. Any two nonempty $\beta\omega$ -closed sets in X are $\beta\omega$ -separated if they are disjoint sets.

Definition 5.3. A topological space (X, τ) is called a $\beta\omega$ -disconnected space if it is the union of two $\beta\omega$ -separated sets. Otherwise (X, τ) is called a $\beta\omega$ -connected space.

Example 5.4. Any a countable topological space (X, τ) is a $\beta\omega$ -disconnected space if X has more than one point. The proof of the following theorem is clear since $Cl_{\beta\omega}(A) \subset Cl(A)$.

Theorem 5.5. Every disconnected space is a $\beta\omega$ -disconnected space. The converse of the above theorem need not be true.

Example 5.6. In the topological space (X, T) , where $T = \{\emptyset, X\}$ and $X = \{a, b\}$, is $\beta\omega$ -disconnected space but it is a connected space.

Theorem 5.7. A topological space (X, τ) is a $\beta\omega$ -disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega$ -open sets.

Proof. Suppose that (X, τ) is a $\beta\omega$ -disconnected space. Then X is the union of two $\beta\omega$ -separated sets, that is, there are two nonempty subsets A and B of X such that

$$Cl_{\beta\omega}(A) \cap B = \emptyset, A \cap Cl_{\beta\omega}(B) = \emptyset \text{ and } A \cup B = X.$$

Take $G = X - Cl_{\beta\omega}(A)$ and $H = X - Cl_{\beta\omega}(B)$. Then G and H are $\beta\omega$ -open sets. Since $B \neq \emptyset$ and $Cl_{\beta\omega}(A) \cap B = \emptyset$, then $B \subseteq X - Cl_{\beta\omega}(A)$, that is, $G = X - Cl_{\beta\omega}(A) \neq \emptyset$. Similar $H \neq \emptyset$. Since $Cl_{\beta\omega}(A) \cap B = \emptyset$, $A \cap Cl_{\beta\omega}(B) = \emptyset$ and $A \cup B = X$, then

$$X - (G \cap H) = (X - G) \cup (X - H) = [Cl_{\beta\omega}(A)] \cup [Cl_{\beta\omega}(B)] = X.$$

That is, $G \cap H = \emptyset$.

Conversely, suppose that (X, τ) is the union of two disjoint nonempty $\beta\omega$ -open subsets, say G and H . Take $A = X - G$ and $B = X - H$. Then A and B are $\beta\omega$ -closed sets, that is, $Cl_{\beta\omega}(A) = A$ and $Cl_{\beta\omega}(B) = B$. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

That is, (X, τ) is a $\beta\omega$ -disconnected space. $\square$

Corollary 5.8. A topological space (X, τ) is a $\beta\omega$ -disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega$ -closed subsets.

Proof. Suppose that (X, τ) is a $\beta\omega$ -disconnected space. Then by Theorem (5.7), (X, τ) is the union of two disjoint nonempty $\beta\omega$ -open subsets, say G and H . Then $X - G$ and $X - H$ are $\beta\omega$ -closed subsets. Since $G \neq \emptyset$, $H \neq \emptyset$ and $X = G \cup H$ then $X - G \neq \emptyset$, $X - H \neq \emptyset$ and

$$(X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Since $G \cap H = \emptyset$ then

$$(X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega$ -closed subsets.

Conversely, suppose that (X, τ) is the union of two disjoint nonempty $\beta\omega$ -closed subsets, say G and H . Take $A = X - G$ and $B = X - H$. Then A and B are $\beta\omega$ -open sets. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Then by Theorem (5.7), (X, τ) is a $\beta\omega$ -disconnected space. $\square$

Theorem 5.9. A topological space (X, τ) is a $\beta\omega$ -connected space if there is no nonempty proper subset of X which is both $\beta\omega$ -open and $\beta\omega$ -closed.

Proof. Suppose that (X, τ) is a $\beta\omega$ -connected space. Let A be a nonempty proper subset of X which is both $\beta\omega$ -open and $\beta\omega$ -closed. Then $X - A$ is a nonempty proper subset of X which is both $\beta\omega$ -open and $\beta\omega$ -closed. Since $A \cup (X - A) = X$, then by Theorem (5.7), X is a $\beta\omega$ -disconnected space and this a contradiction. So there is no nonempty proper subset of X which is both $\beta\omega$ -open and $\beta\omega$ -closed set.

Conversely, suppose that (X, τ) is a $\beta\omega$ -disconnected space. Then by Theorem (5.7), X is the union of two disjoint nonempty $\beta\omega$ -open subsets, say A and B . Then $X - B = A$ is $\beta\omega$ -closed subset of X . Since $B \neq \emptyset$ and $X = A \cup B$ then A is a nonempty proper subset of X which is both $\beta\omega$ -open and $\beta\omega$ -closed. This is a contradiction with the hypothesis. Hence (X, τ) is a $\beta\omega$ -connected space. $\square$

Theorem 5.10. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a $\beta\omega$ -continuous surjection function. If X is a $\beta\omega$ -connected space then Y is connected space.

Proof. Suppose that Y is a disconnected space. Then by Theorem (5.7), Y is the union of two disjoint nonempty open subsets, say G and H . Since f is a $\beta\omega$ -continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega$ -open sets in X . Since $G \neq \emptyset$, $H \neq \emptyset$ and f is a surjection then $f^{-1}(H) \neq \emptyset$ and $f^{-1}(G) \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = Y$ then

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset$$

and

$$f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(Y) = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega$ -open subsets, that is, X is a $\beta\omega$ -disconnected space. This is a contradiction. Hence Y is a connected space. $\square$

Theorem 5.11. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a slightly $\beta\omega$ -continuous surjection function. If X is a $\beta\omega$ -connected space then Y is connected space.

Proof. Suppose that Y is a disconnected space. Then by Theorem (2.2), Y is the union of two disjoint nonempty open subsets, say G and B . Then G and B are clopen sets in Y . Since f is a slightly $\beta\omega$ -continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega$ -open sets in X . Since $G \neq \emptyset$, $H \neq \emptyset$ and f is a surjection then $f^{-1}(H) \neq \emptyset$ and $f^{-1}(G) \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = Y$ then $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Hence X is the union of two disjoint nonempty $\beta\omega$ -open subsets, that is, X is a $\beta\omega$ -disconnected space. This is a contradiction. Hence Y is a connected space. $\square$

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