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USING ADVANCED MATHEMATICAL METHOD TO FIND NORMAL CURVATURE OF THE CURVE ON REGULARSURFACES

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Abstract

Curvature, being a crucial geometric parameter in many applications of surface analysis, is theamount of deviation of curve from a straight line or plane. This study's objective is to use Matlabto objectively calculate Normal Curvature on Smooth Surfaces. The following article concernscurvature and normal curvature with the focus on two uses: first for the manufacturing of automobile tire's inner tubes, and second one to identify the ideal choice degree of normal curvature. The aim of the study is finding the normal curvature numerically and using Matlab, which can carry out complex computational tasks with general commands and is relatively more precise and swifter than the standard numerical calculations.

Keywords: *normal curvature, smooth surface, curve, Matlab*

INTRODUCTION

Calculus is used in the mathematical discipline of differential geometry to examine the geometric characteristics of curves and surfaces. As a consequence of and in relation to the mathematical analysis of curves and surfaces, it emerged and evolved. Mathematicians from the 18th and 19th centuries, primarily Euler (1707–1783), Monge (1746–1818), and Gauss (1777–1855), are the sources of the theory presented in this research. Some of the persistent and unresolved calculus-related concerns, examples include the justifications for the relationships between complex forms, curves, series, and analytical functions, which have beenaddressed through the mathematical study of curves and surfaces. An essential component of differential geometry is the study of curves [1]. The primary issues of differential geometry arethe same as those in Euclidean geometry, employing the techniques of calculus and linear algebra, namely, how to measure lengths, angles, and areas in a more generic context [2]. The core of plane geometry is the analysis of polygonal and spherical properties. Curves that may be locally approximated by straight-line segments are the subject of differential geometry. Thefocus of differential calculus is on functions. In a coordinate system where the horizontal variable influences the vertical variable, these calculus operations can be seen as single-valuedbranches of curves [3]. Calculus and linear algebra are used to explore geometrical problems inthe field of mathematics known as differential geometry. The theory of planes, curves, and surfaces in three-dimensional Euclidean space served as the foundation for differential geometry's development in the 18th and 19th centuries [4]. The concepts of curvature and normalcurvature will be looked at in this subject.

1. CURVES

Definition (1.1): A differentiable curve is a C^{∞}map: $I \to \boldsymbol{R}^3$On has been defined as a vector- valued function that can exist within an open, potentially infinite interval I of R.

The term differentiable indicates that in the function a, there exists a derivative:

$$(t) = (x(t), y(t), z(t)),$$

The differentiability of x, y, and z as coordinate functions holds true. The vector

$$\alpha'(t) = (x'(t), y'(t), z'(t))$$

The derivatives comprise the constituents of which it is composed of the components of a is called the tangent vector, or velocity vector, of the curve α at $t \in I$, or at the point (t), even though this last expression is ambiguous [5].

Theorem (1.2): A reparameterization$\beta$of a such that β it has unit speed exists if is a regular curve in $\boldsymbol{R}^3$.

Proof: Consider the arc length function after fix $S(t) = \int_a^b \|\alpha'(u)\| du.$ ain I of α: $I \to \boldsymbol{R}^3$

It is said that the reparameterization that results is based at $t = a$. The speed function $v = \|\alpha'\|$ of α is the resultant derivative of the function $s = s(t)$. Since α is regular, by definition α' is never zero; hence $\frac{ds}{dt}>0$. According to a well-known calculus theorem, the inverse of the function s is given by the expression $t = t(s)$, twhichis derived at $s = s(t)$ is equal to the opposite of the expression at t = t(s). specifically, $\frac{dt}{ds}>0$.

Let now β be the rectification of α, where $\beta(s) = (t(s))$. We assert thatβ has unit speed.

$$\beta'(s) = \frac{dt}{ds}(s)\ \alpha'\ (t(s))$$

Hence, the speed of b is:

$$||\beta'\ (s)\ || = \frac{dt}{ds}(s)\ ||\ \alpha'\ (t(s))\ || = \frac{dt}{ds}(s)\ \frac{ds}{dt}(t(s)) = 1.$$

Definition (1.3): Each value of t within the interval I corresponds to a tangent vector $Y(t)$ in R^3 at the point α (t), defined by a vector field Y on curve $\alpha : I \to \boldsymbol{R}^3$ [6].

Proposition (1.4): The curvature of a space curve must be zero everywhere in order for it to beconsidered a straight line.

Proof: The standard line equation is as follows: $\alpha(s) = sv + \boldsymbol{c}$ where v is a unit vector and c is aconstant vector. Then $\alpha'(s) = \boldsymbol{T}(s) = \boldsymbol{v}$ is constants o$k = 0$. on the other hand, if $k = 0$ then T$(s) = T0$a constant vector that can be obtained by integrating:

$$(s) = \int_0^s T(u)du + \alpha(0) = s\,T_0 + (0)$$

Here is the line's parametric equation once more [7].

2. SURFACES

Basic Definitions (2.1): The regular surface defined by the R^3 standard, to which we restrict our analysis. Multiple aspects conceal the text's physical significance. In R^3, a smooth, 2- dimensional surface that appears close and is devoid of sharp edges or discontinuities is referredto as a regular surface. Regular surfaces are a particular type of manifold in Euclidean space. As instances of regular surfaces, consider the following: Sphere Plane Any differentiable function's torus and graph, such as the paraboloid $(x, y) = x^2 + y^2)$. These surfaces do not fit thedescription of regular surfaces: Cone (due to the pointed tip) and Line (due to the one- dimensionality and lack of local planarity) [2].

Definition (2.2): If $f: X \rightarrow Y$ is continuous and bijective, and if its inverse map $f^{-1}: Y \rightarrow X$ is alsocontinuous, X and Y are therefore said to be homeomorphic, and f is referred to as a homeomorphism [8].

THE GEOMETRY OF SURFACES

Once we understand how a surface works, determining its shape becomes an important challenge. Surface geometry detection is one of the fundamental mathematical techniques thatis of paramount importance, as it is in all of the natural sciences. This refers to the following. We understand that the given nonlinear or curved item is too complex to analyze directly, so we make an approximation using something linear, such as a line, a plane, or an Euclidean space. Then, we investigate the linear item and deduce inferences about the original curved thing from it. Naturally, this is what we really do when we join the Frenet frame to a curve. This method is also helpful in a variety of areas, including algebraic topology and differential equations [9].

SMOOTH SURFACES

Analysis is used in differential mathematics to analyze surfaces and other geometric objects. For example, we should have the choice to understand why a capacity on a surface is differentiable. We must consider a surface with some additional design for this. First, it can beproven that a guide's smoothness $f: U \rightarrow R^n$ is true if all n portions of f, which are capabilities $U \rightarrow R$, have persistent midway subordinates of all orders. This is predicated on the assumptionthat U is an open subset of R^m. The passage's fractional subsidiaries were then registered according to components.

Definition (2.3): If a surface patch: $\sigma: U \rightarrow R^3$ is smooth, and the vectors u and v are linearly independent at all places $(u, v) \in U$ the patch is said to be regular. Equivalently, $\sigma u \times \sigma v$ shouldnot be 0 at any point along U in order for σ the vector product to be done smoothly [10].

3. CURVATURE

The curvature of a curve indicates how far it deviates from a straight line. The curve does not change direction and is therefore a straight line if the tangent T and the normal N remain constant along the curve. We define curvature as the rate of change of T or N as a result of this.Again, the rate of change is calculated per unit arclength to confirm that the curvature is a geometric quantity for an unparametrized (oriented)curve.

The statement has a derivative worth noting $|T|^2 = \langle T, T \rangle = 1$ results in $2\langle \frac{dT}{ds}, T \rangle = 0$.

What it means is that $\frac{dT}{ds}$ is a scalar multiple of $N$and that the rate of changed $\frac{dT}{ds}$ of T is always orthogonal to T. This scalar multiple is known as the curvature [11].

Definition (3.1): The curvature $\kappa(t)$ of a unit-speed curve with parameter t is defined as $\gamma''(t)$ifis such a curve [10].

Definition (3.2): Let $\boldsymbol{\gamma}$: $I \rightarrow \boldsymbol{R^n}$ be a regular curve. The function of its curvature κ: $I \rightarrow [0,\infty)$ isdefined as:

$$k(t) = \frac{a^{\perp}(t)}{|v(t)|^2} \quad [12]$$

Theorem (3.3): (Meusnier's theorem)

Assume $S \subset R^3$ has a second basic form II and a unit normal field N and is an orientable regularsurface. Let $p \in S$, c:$(-\varepsilon, \varepsilon) \rightarrow S$ S be an arc-length-parameterized curve with $c(0) = p$. Next, there is for the normal curvature k_{nor} of c:

$$k_{nor} = II(c'(0), c'(0))$$

Arc-length parametrized curves, particularly those in S that utilize p, present an important concept. The similarity in the ordinary curvature of tangent vectors further supports our use ofk_{nor} to describe the ordinary curvature of S at that particular point. Reformulated: The only requirement for k_{nor}, other than depending on S and p, is the initial direction of the curve c at point $c'(0)$, and not the specific shape of the curve itself.

Proof: As c lies on S, we have: $\langle N(c(t)), c'(t)\rangle = 0$

for all $t \in (-\varepsilon, \varepsilon)$. By varying this equation, we get: $0 = \frac{d}{dt}\langle N(c(t)), c'(t)\rangle|_{t=0}$

$= \langle N(c(t))_{t=0}, c'(t)\rangle = N(p), c''(t)$

$=\langle d_pN(c'(0)), c'(0)\rangle + k_{nor}$

$=\langle -W_p(c'(0)), c'(0)\rangle + k_{nor}$

$=\langle -II(c'(0), c'(0))\rangle + k_{nor}$ [13]

Definition (3.4): Let $p \in M$ and M be a regular surface in R^3. The major curvatures of M at p, represented by k_1 and k_2, are the highest and lowest values of the normal curvature k of M at p. Principal vectors are unit vectors e_1 and $e_2 \in M_p$ at which these extreme values occur. Head bearings are much the same as other bearings. A bend that follows M and has a digression vector that is necessary everywhere is said to be the main bend on M. The basic curvatures estimate the most notable and least bending points of a typical surface M at each location $p\in$ M. Understanding the curve of a space bend is essential to comprehending the regular ebb andflow.

Lemma (3.5): (Meusnier) Let u_p be a unit tangent vector to M at p, and let β be a unit-speedcurve in M with $\beta(0) = p$ and $\beta'(0) = u_p$. Then

$$k(u_p) = k\,[\beta](0)\cos\theta, (1)$$

where $k[\beta](0)$ is the curvature of β at 0, and θ is the angle between the normal, $N(0)$ of β and the surface normal $U(p)$. Because of this, any curves lying on a surface M thatshare a tangent line at a certain point $p\in M$ will possess similar normal curvature at p.

Proof: Suppose that $k[\beta](0) \neq 0$.we have:

$$k(u_p) = S(u_p).\,u_p = \beta''(0)\,.\,U(p) (2)$$
$$= k\,[\beta]N(0).U(p) = k\,[\beta](0)\cos\theta. (3)$$

In the exceptional case that $k\,[\beta](0) = 0$, the normal $N(0)$ is not defined, butwe still have $k(u_p) = 0$ [14].

Theorem (3.6): A necessary and sufficient condition for the existence of a unique At each place where the osculating plane meets a C^2-regular curve, the curvature will be presentof γ be nonzero at this point [15].

4. NORMAL CURVATURE

Definition (4.1): Let p be a point of $M \subset \boldsymbol{R}^3$. The primary curvatures of M at p are the highestand lowest values of the normal curvature $k(u)$ of M at p, and they are represented by the letters k_1 and k_2, respectively. Are the main curvatures of M at p. The main directions of M at p are thedirections of occurrence of these extreme values. The term "principal vectors of M at p" refersto vectors that are units in these directions.

Definition (4.2): If the normal curvature $k(u)$ is constant across all unit tangent vectors at point p, then $M\subset\boldsymbol{R}^3$ has an umbilic point [6].

Definition (4.3): Consider a regular surface of class C^2 represented by

$$f: U \longrightarrow \mathrm{R}^3, (u,v) \mapsto f(u,v)$$

and a regular curve on this surface with the symbol for class C^2 being,

$$g:]a, b[\longrightarrow U \subseteq \mathrm{R}^2.$$

Write $\bar{h}(s)$ for the normal representation of the curve $h = f \circ g$ on the surface. At the point where the parameter s0 is present, the following amount represents the normal curvature of this curve(with respect to f): $k_n(s_0) = (\vec{\bar{h}}''(s_0) \mid \vec{n}(\bar{h}(s_0)))$
Where $\vec{n}$ indicates the normal vector to the surface [16].

Theorem (4.4): Assume that the $p \in S$ and that S is a regular surface. There is an orthonormalfoundation (e1, e2) of the tangent plane at p such that $dN_p(e_1) = -k_1e_1$ and $dN_p(e_2) = -k_2e_2$. Suppose w.l.o.g that $k_1\leq k_2$. Then, for p, k_1 represents the highest normal curvature and k_2 represents the lowest normal curvature [2].

Example (4.5): Find the normal curvature vector k_n and normal curvature k_n of the curve $u = t^2$, $v = t$ on the surface $x = u\,e_1 + v e_2 + (u^2+v^2)\,e_3$ at the point $t = 1$

SOLUTION:

$x = u\, e_1 + v e_2 + (u^2+v^2)\, e_3$

$x_u = e_1 + 2\, u\, e_3$

$x_{uu} = 2e_3$

$x_v = e_2 + 2\, v\, e_3$

$x_{vv} = 2\, e_3$

$x_{uv} = x_{vu} = 0$

$E = x_u \,.x_u$

$E = 1 + u\, u^2$

$F = x_u \,.x_v = u\, u\, v$

$G = x_v \,.x_v = 1 + u\, v^2$

$$x_u \times x_v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2\, u\, e_1 - 2\, v\, e_2 + e_3$$

$| x_u \times x_v | = \sqrt{u\, u^2 + u\, v^2 + 1}$

As

$$N = \frac{x_u \times x_v}{| x_u \times x_v |} = \frac{-2u\, e_1 - 2v\, e_2 + e_3}{(u\, u^2 + u\, v^2 + 1)^{1/2}}$$

$$L = N \,.x_{uu} = \frac{2}{(u\, u^2 + u\, v^2 + 1)^{1/2}}$$

$M = N.x_{uv} = 0$

$$N = N \,.x_{vv} = \frac{2}{(u\, u^2 + u\, v^2 + 1)^{1/2}}$$

$u = t^2,\ v = t$

$\frac{du}{dt} = 2t, \frac{v}{dt} = 1$

At, $t = 0$

$u = 1\ ,\ v = 1$

$E = 1 + 4 = 5\quad ,\ F = 4\quad ,\ G = 5$

$$N = \frac{-2e_1 - 2\, e_2 + e_3}{(uu^2 + uv^2 + 1)^{1/2}}$$

$N = \frac{-1}{3}(2e_1 + 2\, e_2 - \ )_3$

$L = \frac{2}{3}$, $M = 0$, $N = \frac{2}{3}$

$\frac{du}{dt} = 2,\ \frac{dv}{dt} = 1$

$$k = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}$$

$$= \frac{\frac{2}{3}(u)^2 + 2(0) + \frac{2}{3}(1)^2}{5\,(2)^2 + 2\,(u)(2)(1) + 5\,(1)^2}$$

$$= \frac{\frac{8}{3} + \frac{2}{3}}{20+16+5} \Rightarrow \frac{\frac{10}{3}}{41}$$

$k_n = \frac{10}{123}$

$k_n = k_n N = \quad (2e\ + 2\, e\ - \ e\ ))$

$k_n \ \frac{\ }{369}\ (-2e_1 - 2e_2 + e_3).$

Example 4.6: Find the normal curvature vector k_n and normal curvature k_n of the curve u = t^2, v = t on the surface x = u e_1+ ve_2+(u^2+v^2) e_3 at the point t = 1 (using Matlab).

SOLUTION:

```
clearall
clc
symsxuxudxvfte1e2e3xuuxvvEFGdudtdvNMKnLxuvxvud
x=(u*e1)+v*e2+(u^2 + v^2)*e3;
xu = diff(x,u)
xuu=diff(xu)
xv = diff(x,v)
xvv=diff(xv)
xuv=0
xvu=0
E=xu*xu
u=t^2
v=t
E=1+u*u^2
F=xu*xv
G=xv*xv
G=1+u*v^2
du=diff(u)
dv=diff(v)
N=(xu*xv)/abs(xu*xv)
L=N*xuu
M=N*xuv
N=N*xuu
t=1
u(t)=t^2
v(t)=t
%Kn=(L*(du^2)+2*M(du*dv)+(N*(dv^2)))/(E*(du^2)+2*F(du*dv)+(G*(dv^2)))
Kn=-10/369
[e1,e2] = meshgrid(-2:0.2:2,-2:0.2:2);
e3=0.2;
Kn=Kn*((2*e1)+(2*e2)-e3);
figure
surf(Kn,sin(e1),cos(e2));
xlabel('e1');
ylabel('e2');
zlabel('e3');
```

RESULT:

```
xu = e1 + 2*e3*u
xuu = 2*e3
xv = e2 + 2*e3*v
xvv = 2*e3
xuv = 0
xvu = 0

E = (e1 + 2*e3*u)^2
u = t^2
v = t

E = t^6 + 1
F = (e1 + 2*e3*u)*(e2 + 2*e3*v)
G = (e2 + 2*e3*v)^2
G = t^4 + 1
du = 2*t
dv = 1

N = ((e1 + 2*e3*u)*(e2 + 2*e3*v))/abs((e1 + 2*e3*u)*(e2 + 2*e3*v))
L = (2*e3*(e1 + 2*e3*u)*(e2 + 2*e3*v))/abs((e1 + 2*e3*u)*(e2 + 2*e3*v))
M = 0
N = (2*e3*(e1 + 2*e3*u)*(e2 + 2*e3*v))/abs((e1 + 2*e3*u)*(e2 + 2*e3*v))
t = 1
u = 1
v = 1
Kn = -0.0271
```

REPRESENT THE SOLUTION GRAPHICALLY:

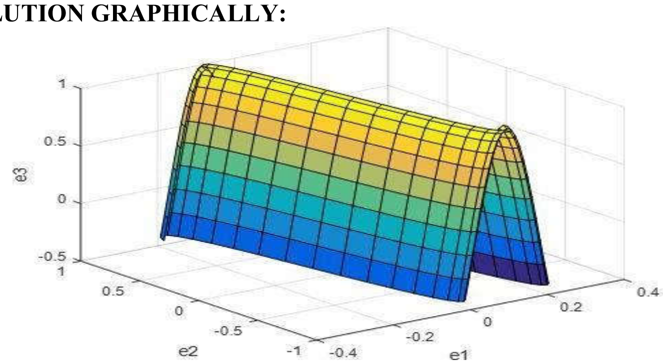

Figure 5.1: Normal Curvature Vector on the Surface at the Point t=1

RESULTS

Following our demonstration of a novel mathematical approach for calculating normal curvature in cylindrical coordinates, we discovered the following findings. Finally, we may consider the new mathematical approach as a theory, which is regarded as one of the most accurate theories, having described the feasibility of the calculation of normal curvature by a new mathematical technique with a very high rate and accuracy.

CONCLUSION

The following discoveries were discovered because of our presentation of a novel mathematicaltechnique for determining normal curvature in cylindrical coordinates: Finally, after describingthe viability of the calculation of normal curvature by a novel, highly accurate and speedy mathematical technique, we may take the new mathematical technique into consideration as a theory that is recognized as among the most correct theories.

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