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A TWO-ECHELON SUPPLY CHAIN MODEL FOR TIME DEPEPENDENT DETERIORATING PRODUCTS WITH TIME-DEPENDENT DEMAND, DEMAND-DEPENDENT PRODUCTION RATE AND SHORTAGES WITH FUZZY AND HEXAGONAL FUZZY DEMANDS

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Abstract

In this paper, we have cultivated a two- stage supply chain production inventory model for time dependent deteriorating product with time-dependent demand with demand dependent production. Shortages are allowed only in the retailer's inventory. This model is framed for a definite period of time horizon. In reality, it is often found that if the manufacturer fabricated a huge amount of product, it may experience holding cost and other costs also. And moreover it may so happen that the items may get damaged or deteriorated during this time period which will lead to incurring the cost for deterioration. On the other hand, an inadequate amount of product may lead to inventory shortage alongwith penalty cost. Therefore to bring stability over these two unwanted situations, the manufacturer will fix the rate of production as demand dependent. Henceforth the production rate is assumed to be a function of demand rate, which is time dependent. A numerical example is shown and sensitivity analysis with respect to different associated parameter is also depicted to study the case.

Keywords: *Inventory, Fuzzy and Hexagonal Fuzzy Demand.*

INTRODUCTION

We developed a model for inventory management in a two-stage supply chain, taking into account time-dependent deterioration of the product and time-dependent demand that influences production. Shortages are allowed only in the retailer's inventory. This model is framed for a definite period of time horizon. In reality, it is often found that if the manufacturer fabricated a huge amount of product, it may experience holding cost and other costs also. And moreover it may so happen that the items may get damaged or deteriorated during this time period which will lead to incurring the cost for deterioration. On the other hand, an inadequate amount of product may lead to inventory shortage alongwith penalty cost. Therefore to bring stability over these two unwanted situations, the manufacturer will fix the rate of production as demand dependent. The production rate is now considered to depend on the demand rate, which varies over time. An illustrative example is provided, and sensitivity analysis is carried out to examine the impact of various related parameters on the case.

A supply chain is a network of different stages, such as supplier, manufacturer, distributor and retailer that performs the functions of procurement of materials, transformation of these materials into intermediate and finished products and distribution of these finished products to customers. In traditional business environment, each business tries to minimize its cost only without considering the cost of the other supply chain members, although most of the decisions made by any of these stages have direct and indirect impacts on the profitably of the other members. So, the assumption that can greatly influence the optimal policies of any business organization is to take a supply chain perspective when analysing pricing models. To date, very few studies on deteriorating inventory in two-stage supply chain systems have been carried out. Fuzzy set theory has been applied to several fields like optimization and other areas. In the field of research, L. A. Zadeh [1] developed the fuzzy set theory which is used in several research in different fields like mathematics, statistics. In 2014, Chang [2] framed a model of production inventory with deterioration items in a two echelon supply chain. In 2014, Tayal et. al. [3] modelled the supply chain model for deteriorating items involves two echelons and includes effective investment in preservation technology. In 2017, Saha and Chakrabarti [4] schemed A model for managing inventory of deteriorating goods with demand that depends on price, using fuzzy logic, within a supply chain. In 2022, Sen and Chakrabarti [5] developed An Industrial Production Inventory model with deterioration under neutrosophic fuzzy optimization. In 2017, Teimoury et. al. [6] framed An Integrated Pricing and Inventory Model for Deteriorating Products in a two stage supply chain under Replacement and Shortage. In 2014, Jaggi et. al [7] framed a the model for managing inventory in two warehouses for items that deteriorate, allowing for a delay in payment with partial backlogging. Sen and Chakrabarti [8] developed An EOQ model for healthcare industries with exponential demand pattern and time dependent elayed deterioration under fuzzy and neutrosophic environment. Chakrabarti and Chaudhuri [9] developed The EOQ model deals with deteriorating items experiencing a linear trend in demand and shortages occurring in all cycles. In 2008, Zhou et. al. [10] developed A Supply-chain coordination under an inventory-level-dependent demand rate. Shee and Chakrabarti [11] put forward The supply chain system has a fuzzy inventory model for items that deteriorate, and the demand rate depends on time. In 2011, Jaggi and Kauser [12] framed When demand is influenced by price and credit period, a retailer's ordering policy in a supply chain is impacted.On top of all, In 2018, Saha and Chakrabarti [13] framed a inventory A Two-Echelon Supply Chain Model for Deteriorating Product with Time-Dependent Demand, Demand-Dependent Production Rate and Shortage. In 2023, Sen and Chakrabarti [14] proposed with an idea of An EOQ model for time dependent deteriorating items with octagonal fuzzy Quadratic demand. In 2010, Shah et. al. [15] An approach that integrates optimal unit price and credit period for a deteriorating inventory system is presented in the study, considering price-sensitive buyer demand. In 2023, Sen and Chakrabarti [16] upheld an inventory model named An EPQ model for products with two parameter Weibull distribution deterioration with fuzzy and decagonal fuzzy demand. In 2013 Liao et. al. [17] modeled A deterministic inventory model designed for perishable goods within a supply chain framework, incorporating two storage facilities and trade credit considerations. Also, Wang, Tang and Zhao [18] developed model on fuzzy economic order quantity inventory model without backordering.

Here we discuss the basics of Fuzzy, mainly triangular fuzzy and hexagonal fuzzy with defuzzification under signed distance method.

Basic Preliminaries:

A Fuzzy Set A is defined by a membership function $\mu_A(x)$ which maps each and every element of X to [0, 1]. i.e. $\mu_A(x) \rightarrow [0,1]$, where X is the underlying set. In simple, a fuzzy set is a set whose boundary is not clear. On the other hand, a fuzzy set is a set whose element are characterized by a membership function as above.

A triangular fuzzy number is a fuzzy set. It is denoted by A = <a, b, c> and is defined by the following membership function:

$$\pi_A(x) = \begin{cases} 0 & x \le a \\ (x-a)/(b-a) & a \le x \le b \\ (c-x)/(c-b) & b \le x \le c \\ 0 & x \ge c \end{cases}$$

Defuzzification of Triangular fuzzy number

Defuzzification, i.e. Signed distance for A = <a, b, c>, a triangular fuzzy number, the signed distance of A measured from O_1 is given by

$$d(A, O_1) = \tfrac{1}{4}(a + 2b + c)$$

Hexagonal Fuzzy Number

A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is a hexagonal fuzzy number, if its membership function $\mu_{\tilde{A}(x)}$ is

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 \ , & x < a_1 \\ \frac{1}{2}\left(\frac{x-a_1}{a_2-a_1}\right) \ , & a_1 \le x \le a_2 \\ \frac{1}{2}+\frac{1}{2}\left(\frac{x-a_2}{a_3-a_2}\right) \ , & a_2 \le x \le a_3 \\ 1 \ , & a_3 \le x \le a_4 \\ 1-\frac{1}{2}\left(\frac{x-a_4}{a_5-a_4}\right) \ , & a_4 \le x \le a_5 \\ 0 \ , & a_6 \le x \end{cases}$$

Defuzzification of a Hexagonal fuzzy number:

Defuzzification of a hexagonal fuzzy number by Graded mean integration method is given by

$A^0 = \frac{1}{12}(a_1 + 3a_2 + 2a_3 + 2a_4 + 3a_5 + a_6)$

Assumptions and Notations:

1. The rate of deterioration is time dependent.
2. Time horizon is finite.
3. The demand for the products is time dependent.
4. The production rate is dependent on demand rate.
5. The shortages are allowed for retailers only
6. The demand rate function D(t) is assumed to be a function of time in a polynomial form: $D(t) = \beta t^{\gamma-1}$, with $\gamma > 1$ for increasing demand, $\gamma < 1$ for decreasing demand and $\gamma = 1$ for constant demand and β is a positive constant.
7. T = Time horizon
8. K = Production coefficient > 1
9. θ is the rate of deterioration, $0 < \theta < 1$
10. t_1 = production period for manufacturer.
11. C_m = production cost per unit for the manufacturer.
12. h_m = holding cost per unit for the manufacturer
13. h_r = holding cost per unit for the retailer
14. A_m = set up cost per production run for the manufacturer.
15. s = shortage cost per unit for the retailer
16. Q = initial inventory level for the retailer
17. p = purchasing cost per unit for the retailer
18. r = the time at which inventory level becomes zero for the retailer.
19. d_m = cost for deterioration for the manufacturer.
20. d_r = cost for deterioration for the retailer.

Mathematical Model:

Case 1: When demand is Crisp:

Manufacturer's model:

The manufacturer starts his production process at time t = 0 and continues up to time $t = t_1$, where the inventory level reaches its maximum level. Production then stops at $t = t_1$ and the inventory gradually depletes to zero at the end of cycle time t = T due to demand and deterioration.

The variation in inventory levels can be represented by the subsequent differential equation.

$I'(t) + \theta t I(t) = (K-1)\beta t^{\gamma-1} \qquad 0 \le t \le t_1$

$I'(t) + \theta t I(t) = -\beta t^{\gamma-1} \qquad t_1 \le t \le T$

Satisfying the boundary condition I(0) = 0, I(T) = 0

The solution of this system is given by

$$I(t) = (K-1)\beta\left(\frac{t^\gamma}{\gamma} - \frac{\theta t^{\gamma+2}}{\gamma(\gamma+2)}\right) \qquad 0 \le t \le t_1$$

$$I(t) = \frac{-\beta}{\gamma}\left[(t^\gamma - T^\gamma) - \frac{\theta\gamma}{2(\gamma+2)}(t^{\gamma+2} + T^{\gamma+2}) - \frac{\theta}{2}(t^{\gamma+2} - T^\gamma t^2)\right] \qquad t_1 \le t \le T$$

Production cost (PC) $= C_m \int_0^{t_1} K\beta t^{\gamma-1}\, dt$

$= \frac{C_m K \beta t_1^\gamma}{\gamma}$

Holding cost (HC) $= h_m\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt\right]$

$$= h_m\left[(K-1)\beta\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\beta}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right]$$

Deterioration cost (DC) $= d_m\left[\int_0^{t_1}\theta t I(t)dt+\int_{t_1}^{T}\theta t I(t)dt\right]$

$$= d_m\left[(K-1)\,\theta\,\beta\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\beta\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{T^{\gamma}}{2}\left(T^2-t_1^2\right)\right]\right]$$

So, the total cost of the manufacturer is given by

$TC_m = A_m + PC + HC + DC$

$$= A_m+\frac{C_m K\beta t_1^{\gamma}}{\gamma}+h_m\left[(K-1)\beta\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\beta}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right]+d_m\left[(K-1)\,\theta\,\beta\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\beta\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{T^{\gamma}}{2}\left(T^2-t_1^2\right)\right]\right]$$

Retailer's model

The inventory cycle commences at time t = 0. Throughout the interval $[0, t_1]$, the inventory level diminishes as a result of demand and deterioration. At time $t = t_1$, the inventory level reaches zero, leading to shortages that persist until time t = T.

The differential equations showing the behavior of the system are given as follow:

$I'(t) + \theta t I(t) = -\beta t^{\gamma-1}$ $\quad 0 \le t \le t_r$

$I'(t) = -\beta t^{\gamma-1}$ $\quad t_r \le t \le T$

Satisfying the boundary conditions, $I(0) = Q$, $I(t_r) = 0$, $I(T) = -S$

The solution of the above differential equation is given by:

$$I(t) = -\beta\left[\frac{t^{\gamma}-t_1^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}\left(t^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{\theta}{2\gamma}\left(t^{\gamma+2}-t^2t_1^{\gamma}\right)\right] \quad 0 \le t \le t_r$$

$$I(t) = \frac{-\beta}{\gamma}\left(t^{\gamma}-t_1^{\gamma}\right) \quad t_r \le t \le T$$

Now I(0) = Q, gives

$$Q = \beta\left(\frac{t_r^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right)$$

Now, Purchasing cost (PC) $= Q.p$

$$= \beta p\left(\frac{t_r^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right)$$

Holding cost (HC) $= h_r\int_0^{t_1} I(t)dt$

$$= -h_r\beta\left(\frac{-t_r^{\gamma+1}}{\gamma+1}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3}+\frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right)$$

Deterioration cost (DC) $= d_r\int_0^{t_1}\theta t I(t)dt$

$$= \frac{-\beta\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)}-\frac{t_1^{\gamma}t_r^{2}}{2}\right]$$

Shortage cost (SC) $= -s\int_{t_1}^{T} I(t)dt$

$$= -s\frac{\beta}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1}-Tt_r^{\gamma}+\frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$$

Therefore total cost of the retailer is given by

$TC_r = PC + HC + DC + SC$

$$= \beta p\left(\frac{t_r^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right)-h_r\beta\left(\frac{-t_r^{\gamma+1}}{\gamma+1}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3}+\frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right)+\frac{-\beta\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)}-\frac{t_1^{\gamma}t_r^{2}}{2}\right]-s\frac{\beta}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1}-Tt_r^{\gamma}+\frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$$

Hence the total cost of the entire supply chain is given by

$TC = TC_m + TC_r$

$$= A_m+\frac{C_m K\beta t_1^{\gamma}}{\gamma}+h_m\left[(K-1)\beta\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\beta}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right]+d_m\left[(K-1)\,\theta\,\beta\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\beta\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\right.\right.$$

$\frac{T^{\gamma}}{2}(T^2 - t_1^2)\Big]\Big]$ + $\beta p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right)$ - $h_r\beta\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+3} \frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)} t_r^{\gamma+3}\right)$ + $\frac{-\beta\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma} t_r^2}{2}\right]$ - $s\frac{\beta}{\gamma}$ $\left(\frac{T^{\gamma+1}}{\gamma+1} - T t_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

Case 2: When demand is fuzzy

Manufacturer's model:

The manufacturer starts his production process at time t = 0 and continues up to time $t = t_1$, where the inventory level reaches its maximum level. Production then stops at $t = t_1$ and the inventory gradually depletes to zero at the end of cycle time t = T due to fuzzy demand and deterioration.

The variation in inventory levels can be represented by the subsequent differential equation.

$I'(t) + \theta t I(t) = (K-1)\tilde{\beta} t^{\gamma-1}$ $\quad 0 \le t \le t_1$

$I'(t) + \theta t I(t) = -\tilde{\beta} t^{\gamma-1}$ $\quad t_1 \le t \le T$

With boundary condition I(0) = 0, I(T) = 0

The solution of this system is given by

$I(t) = (K-1)\tilde{\beta}\left(\frac{t^{\gamma}}{\gamma} - \frac{\theta t^{\gamma+2}}{\gamma(\gamma+2)}\right)$ $\quad 0 \le t \le t_1$

$I(t) = \frac{-\tilde{\beta}}{\gamma}\left[(t^{\gamma} + T^{\gamma}) - \frac{\theta}{(\gamma+2)}(t^{\gamma+2} + T^{\gamma+2})\right]$ $\quad t_1 \le t \le T$

Production cost (PC) $= C_m\int_0^{t_1} K\tilde{\beta} t^{\gamma-1}\,dt$

$= \frac{C_m K\tilde{\beta} t_1^{\gamma}}{\gamma}$

Holding cost (HC) $= h_m\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt\right]$

$= h_m\Bigg[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1} - \frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right) - \frac{\tilde{\beta}}{\gamma}\Bigg[\frac{1}{\gamma+1}\left(T^{\gamma+1} - t_1^{\gamma+1}\right) - T^{\gamma}(t_2 - t_1) - \frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3} - t_1^{\gamma+3}}{\gamma+3} + T^{\gamma+2}(T - t_1)\right) - \frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3} - t_1^{\gamma+3}\right) - \frac{T^{\gamma}}{3}(T^3 - t_1^3)\right)\Bigg]\Bigg]$

Deterioration cost (DC) $= d_m\left[\int_0^{t_1} \theta t I(t)dt + \int_{t_1}^{T} \theta t I(t)dt\right]$

$= d_m\left[(K-1)\,\theta\,\tilde{\beta}\,\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)} - \frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2} - t_1^{\gamma+2}\right) - \frac{T^{\gamma}}{2}(T^2 - t_1^2)\right]\right]$

So, the total cost of the manufacturer is given by

$TC_m^{+} = A_m + PC + HC + DC$

$= A_m + \frac{C_m K\beta t_1^{\gamma}}{\gamma} + h_m\Bigg[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1} - \frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right) - \frac{\tilde{\beta}}{\gamma}\Bigg[\frac{1}{\gamma+1}\left(T^{\gamma+1} - t_1^{\gamma+1}\right) - T^{\gamma}(t_2 - t_1) - \frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3} - t_1^{\gamma+3}}{\gamma+3} + T^{\gamma+2}(T - t_1)\right) - \frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3} - t_1^{\gamma+3}\right) - \frac{T^{\gamma}}{3}(T^3 - t_1^3)\right)\Bigg]\Bigg] + d_m\left[(K-1)\,\theta\,\tilde{\beta}\,\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)} - \frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2} - t_1^{\gamma+2}\right) - \frac{T^{\gamma}}{2}(T^2 - t_1^2)\right]\right]$

Retailer's model

The inventory cycle commences at time t = 0. Throughout the interval $[0, t_1]$, the inventory level diminishes as a result of fuzzy demand and deterioration. At time $t = t_1$, the inventory level reaches zero, leading to shortages that persist until time t = T.

The differential equations showing the behavior of the system are given as follow:

$I'(t) + \theta t I(t) = -\tilde{\beta} t^{\gamma-1}$ $\quad 0 \le t \le t_r$

$I'(t) = -\tilde{\beta}$ $\quad t_r \le t \le T$

With the boundary conditions, I(0) = Q, $I(t_r) = 0$, I(T) = -S

The solution of the above differential equation is given by:

$I(t) = -\tilde{\beta}\left[\frac{t^{\gamma} - t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)}\left(t^{\gamma+2} - t_r^{\gamma+2}\right) - \frac{\theta}{2\gamma}\left(t^{\gamma+2} - t^2 t_r^{\gamma}\right)\right]$ $\quad 0 \le t \le t_r$

$I(t) = \frac{-\tilde{\beta}}{\gamma}\left(t^{\gamma} - t_r^{\gamma}\right)$ $\quad t_r \le t \le T$

Now I(0) = Q, gives

$Q = \tilde{\beta}\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right)$

Now, Purchasing cost (PC) = Q.p

$= \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right)$

Holding cost (HC) $= h_r \int_0^{t_1} I(t)dt$

$= - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+3} \frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)} t_r^{\gamma+3}\right)$

Deterioration cost (DC) $= d_r \int_0^{t_1} \theta t I(t)dt$

$= \frac{-\tilde{\beta}\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma} t_r^2}{2}\right]$

Shortage cost (SC) $= -s\int_{t_1}^{T} I(t)dt$

$= -s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

Therefore total cost of the retailer is given by

$TC_r^{+} = PC + HC + DC + SC$

$= \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right) - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+3} \frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)} t_r^{\gamma+3}\right) + \frac{\tilde{\beta}\theta}{\gamma(\gamma+1)} t_r^{\gamma+1} - s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

Hence the total cost of the entire supply chain is given by

$TC^{+} = TC_m + TC_r$

$$= A_m + \frac{C_m K\tilde{\beta}t_1^{\gamma}}{\gamma} + h_m\left[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1} - \frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right) - \frac{\tilde{\beta}}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1} - t_1^{\gamma+1}\right) - T^{\gamma}(t_2 - t_1) - \frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3} - t_1^{\gamma+3}}{\gamma+3} + T^{\gamma+2}(T - t_1)\right) - \frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3} - t_1^{\gamma+3}\right) - \frac{T^{\gamma}}{3}(T^3 - t_1^3)\right)\right]\right] + d_m\left[(K-1)\,\theta\,\tilde{\beta}\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)} - \frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2} - t_1^{\gamma+2}\right) - \frac{T^{\gamma}}{2}(T^2 - t_1^2)\right]\right] + \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right) - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+3} \frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)} t_r^{\gamma+3}\right) + \frac{-\tilde{\beta}\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma} t_r^2}{2}\right] - s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$$

After Defuzzification,

$$TC^{+} = A_m + \frac{C_m(\beta_1+2\beta_2+\beta_3)Kt_1^{\gamma}}{4\gamma} + h_m\left[(K-1)\frac{(\beta_1+2\beta_2+\beta_3)}{4}\left(\frac{t_1^{\gamma+1}}{\gamma+1} - \frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right) - \frac{(\beta_1+2\beta_2+\beta_3)}{4\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1} - t_1^{\gamma+1}\right) - T^{\gamma}(t_2 - t_1) - \frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3} - t_1^{\gamma+3}}{\gamma+3} + T^{\gamma+2}(T - t_1)\right) - \frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3} - t_1^{\gamma+3}\right) - \frac{T^{\gamma}}{3}(T^3 - t_1^3)\right)\right]\right] + d_m\left[(K-1)\,\theta\,(\beta_1+2\beta_2+\beta_3)\frac{t_1^{\gamma+2}}{4\gamma(\gamma+1)} - \frac{(\beta_1+2\beta_2+\beta_3)\theta}{4\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2} - t_1^{\gamma+2}\right) - \frac{T^{\gamma}}{2}(T^2 - t_1^2)\right]\right] + \frac{(\beta_1+2\beta_2+\beta_3)}{4}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+2}\right) - h_r\frac{(\beta_1+2\beta_2+\beta_3)}{4}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)} t_r^{\gamma+3} \frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)} t_r^{\gamma+3}\right) + \frac{-(\beta_1+2\beta_2+\beta_3)\theta}{4\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma} t_r^2}{2}\right] - s\frac{(\beta_1+2\beta_2+\beta_3)}{4\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$$

Case 3: When demand is hexagonal fuzzy

Manufacturer's model:

The manufacturer starts his production process at time t = 0 and continues up to time $t = t_1$, where the inventory level reaches its maximum level. Production then stops at $t = t_1$ and the inventory gradually depletes to zero at the end of cycle time t = T due to hexagonal demand and deterioration.

The variation in inventory levels can be represented by the subsequent differential equation.

$I'(t) + \theta t I(t) = (K-1)\,\tilde{\beta}t^{\gamma-1}$ $\quad 0 \le t \le t_1$

$I'(t) + \theta t I(t) = -\tilde{\beta}t^{\gamma-1}$ $\quad t_1 \le t \le T$

With boundary condition I(0) = 0, I(T) = 0

The solution of this system is given by

$I(t) = (K-1)\,\tilde{\beta}\left(\frac{t^{\gamma}}{\gamma} - \frac{\theta t^{\gamma+2}}{\gamma(\gamma+2)}\right)$ $\quad 0 \le t \le t_1$

$I(t) = \frac{-\tilde{\beta}}{\gamma}\left[(t^{\gamma} + T^{\gamma}) - \frac{\theta}{(\gamma+2)}(t^{\gamma+2} + T^{\gamma+2})\right]$ $\quad t_1 \le t \le T$

Production cost (PC) $= C_m\int_0^{t_1} K\tilde{\beta}t^{\gamma-1}\,dt$

$= \frac{C_m K\tilde{\beta}t_1^{\gamma}}{\gamma}$

Holding cost (HC) $= h_m\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt\right]$

$= \quad h_m\left[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\tilde{\beta}}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right]$

Deterioration cost (DC) $= d_m\left[\int_0^{t_1}\theta t I(t)dt+\int_{t_1}^{T}\theta t I(t)dt\right]$

$= d_m\left[(K-1)\,\theta\,\tilde{\beta}\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{T^{\gamma}}{2}\left(T^2-t_1^2\right)\right]\right]$

So, the total cost of the manufacturer is given by

$TC_m^{++} = A_m + PC + HC + DC$

$= A_m + \frac{C_m K\tilde{\beta}t_1^{\gamma}}{\gamma} + h_m\left[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\tilde{\beta}}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right] + d_m\left[(K-1)\,\theta\,\tilde{\beta}\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{T^{\gamma}}{2}\left(T^2-t_1^2\right)\right]\right]$

Retailer's model

The inventory cycle commences at time t = 0. Throughout the interval $[0, t_1]$, the inventory level diminishes as a result of hexagonal fuzzy demand and deterioration. At time $t = t_1$, the inventory level reaches zero, leading to shortages that persist until time t = T.

The differential equations showing the behavior of the system are given as follow:

$I'(t) + \theta t I(t) = -\tilde{\beta}t^{\gamma-1} \qquad 0 \le t \le t_1$

$I'(t) = -\tilde{\beta} \qquad t_1 \le t \le T$

With the boundary conditions, $I(0) = Q$, $I(t_1) = 0$, $I(T) = -S$

The solution of the above differential equation is given by:

$I(t) = -\tilde{\beta}\left[\frac{t^{\gamma}-t_1^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}\left(t^{\gamma+2}-t_1^{\gamma+2}\right)-\frac{\theta}{2\gamma}\left(t^{\gamma+2}-t^2t_1^{\gamma}\right)\right] \qquad 0 \le t \le t_r$

$I(t) = \frac{-\tilde{\beta}}{\gamma}\left(t^{\gamma}-t_1^{\gamma}\right) \qquad t_r \le t \le T$

Now I(0) = Q, gives

$Q = \tilde{\beta}\left(\frac{t_r^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right)$

Now, Purchasing cost (PC) $= Q.p$

$= \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right)$

Holding cost (HC) $= h_r\int_0^{t_1} I(t)dt$

$= -h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3}+\frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right)$

Deterioration cost (DC) $= d_r\int_0^{t_1}\theta t I(t)dt$

$= \frac{-\tilde{\beta}\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)}-\frac{t_1^{\gamma}t_r^2}{2}\right]$

Shortage cost (SC) $= -s\int_{t_1}^{T} I(t)dt$

$= -s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1}-Tt_r^{\gamma}+\frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

Therefore total cost of the retailer is given by

$TC_r^{++} = PC + HC + DC + SC$

$= \tilde{\beta}p\left(\frac{t_1^{\gamma}}{\gamma}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right) - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1}+\frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3}+\frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right) + \frac{-\tilde{\beta}\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)}-\frac{t_1^{\gamma}t_r^2}{2}\right] - s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1}-Tt_r^{\gamma}+\frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

Hence the total cost of the entire supply chain is given by

$TC^{++} = TC_m + TC_r$

$= A_m + \frac{C_m K\tilde{\beta}t_1^{\gamma}}{\gamma} + h_m\left[(K-1)\tilde{\beta}\left(\frac{t_1^{\gamma+1}}{\gamma+1}-\frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right)-\frac{\tilde{\beta}}{\gamma}\left[\frac{1}{\gamma+1}\left(T^{\gamma+1}-t_1^{\gamma+1}\right)-T^{\gamma}(t_2-t_1)-\frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3}-t_1^{\gamma+3}}{\gamma+3}+T^{\gamma+2}(T-t_1)\right)-\frac{\theta}{2}\left(\frac{1}{\gamma+3}\left(T^{\gamma+3}-t_1^{\gamma+3}\right)-\frac{T^{\gamma}}{3}\left(T^3-t_1^3\right)\right)\right]\right] + d_m\left[(K-1)\,\theta\,\tilde{\beta}\frac{t_1^{\gamma+2}}{\gamma(\gamma+1)}-\frac{\tilde{\beta}\theta}{\gamma}\left[\frac{1}{\gamma+2}\left(T^{\gamma+2}-t_1^{\gamma+2}\right)-\right.\right.$

$\frac{T^{\gamma}}{2}(T^2 - t_1^2)\Big]\Big] + \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right) - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right) + \frac{\tilde{\beta}\theta}{\gamma(\gamma+1)}t_r^{\gamma+1} - s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right) + \tilde{\beta}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right) - h_r\tilde{\beta}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right) + \frac{-\tilde{\beta}\theta}{\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma}t_r^2}{2}\right] - s\frac{\tilde{\beta}}{\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$

After Defuzzification,

$$TC^{++} = A_m + \frac{C_m K(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)t_1^{\gamma}}{12\gamma} + h_m\Bigg[(K-1)\frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12}\left(\frac{t_1^{\gamma+1}}{\gamma+1} - \frac{\theta t_1^{\gamma+3}}{\gamma(\gamma+2)(\gamma+3)}\right) - \frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12\gamma}\Bigg[\frac{1}{\gamma+1}(T^{\gamma+1} - t_1^{\gamma+1}) - T^{\gamma}(t_2 - t_1) - \frac{\theta\gamma}{2(\gamma+2)}\left(\frac{T^{\gamma+3} - t_1^{\gamma+3}}{\gamma+3} + T^{\gamma+2}(T - t_1)\right) - \frac{\theta}{2}\left(\frac{1}{\gamma+3}(T^{\gamma+3} - t_1^{\gamma+3}) - \frac{T^{\gamma}}{3}(T^3 - t_1^3)\right)\Bigg]\Bigg] + d_m\Bigg[(K-1)\,\theta\,(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)\frac{t_1^{\gamma+2}}{12\gamma(\gamma+1)} - \frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)\theta}{12\gamma}\left[\frac{1}{\gamma+2}(T^{\gamma+2} - t_1^{\gamma+2}) - \frac{T^{\gamma}}{2}(T^2 - t_1^2)\right]\Bigg] + \frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right) - h_r\frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right) + \frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)\theta}{12\gamma(\gamma+1)}t_r^{\gamma+1} - s\frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right) + \frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12}p\left(\frac{t_r^{\gamma}}{\gamma} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+2}\right) - h_r\frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12}\left(\frac{-t_r^{\gamma+1}}{\gamma+1} + \frac{\theta}{2(\gamma+2)}t_r^{\gamma+3}\frac{\gamma+4}{\gamma+3} + \frac{\theta}{6(\gamma+3)}t_r^{\gamma+3}\right) + \frac{-(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)\theta}{12\gamma}\left[\frac{t_r^{\gamma+2}}{(\gamma+2)} - \frac{t_1^{\gamma}t_r^2}{2}\right] - s\frac{(\beta_1+3\beta_2+2\beta_3+2\beta_4+3\beta_5+\beta_6)}{12\gamma}\left(\frac{T^{\gamma+1}}{\gamma+1} - Tt_r^{\gamma} + \frac{\gamma t_r^{\gamma+1}}{\gamma+1}\right)$$

Problem and Solution Procedure:
The problem is to minimize TC, TC^+, TC^{++}
Here we use Lingo Software for optimization.

Illustrative Example:
$A_m = 80$, $C_m = 8$, $h_m = 3$, $d_m = 7$, $K = 6$, $\gamma = 0.75$, $\theta = 0.05$, $h_r = 2.5$, $d_r = 6.5$, $p = 20$, $s = 15$, $T = 10$
For crisp model, $\beta = 6$, gives TC = 2096.78 at $t_1 = 4.38$, $t_r = 5.67$
For fuzzy model, $\tilde{\beta} = (5.1, 6, 7.2)$, gives $TC^+ = 2085.24$ at $t_1^+ = 4.35$, $t_r^+ = 5.58$
For octagonal fuzzy model, $\tilde{\beta} = (4.5, 5.1, 6, 6.5, 7.2, 7.9)$, gives $TC^{++} = 2077.46$ at $t_1^{++} = 4.22$, $t_r^{++} = 5.41$

Sensitivity Analysis
Sensitivity on θ

θ	t_1	t_r	TC
0.03	4.15	5.33	2078.45
0.04	4.29	5.48	2083.69
0.06	4.51	5.82	2102.23
0.07	4.64	5.99	2109.74

θ	t_1^+	t_r^+	TC^+
0.03	4.11	5.30	2068.22
0.04	4.26	5.43	2079.05
0.06	4.49	5.79	2091.49
0.07	4.62	5.95	2098.37

θ	t_1^{++}	t_r^{++}	TC^{++}
0.03	4.07	5.28	2058.75
0.04	4.17	5.39	2073.96
0.06	4.31	5.75	2087.34
0.07	4.40	5.91	2092.03

Sensitivity on T

T	t_1	t_r	TC
8	4.16	5.31	2072.21
9	4.27	5.47	2079.99
11	4.50	5.80	2096.53
12	4.61	5.97	2104.30

T	t_1^+	t_r^+	TC^+
8	4.09	5.28	2061.27
9	4.23	5.41	2071.69
11	4.48	5.77	2082.08
12	4.59	5.91	2090.73

T	t_1^{++}	t_r^{++}	TC^{++}
8	4.03	5.23	2049.63
9	4.15	5.38	2070.06
11	4.28	5.74	2081.31
12	4.35	5.87	2081.58

Sensitivity on γ

γ	t_1	t_r	TC
0.55	4.14	5.33	2110.23
0.65	.4.28	5.48	2103.39
0.85	4.50	5.82	2081.67
0.95	4.64	5.98	2077.37

γ	t_1^+	t_r^+	TC^+
0.55	4.10	5.30	2099.24
0.65	4.24	5.42	2092.77
0.85	4.50	5.79	2077.66
0.95	4.62	5.94	2064.30

γ	t_1^+	t_r^{++}	TC^{++}
0.55	4.07	5.32	2097.68
0.65	4.17	5.38	2089.07
0.85	4.32	5.75	2075.91
0.95	4.41	5.92	2059.73

Here we observe that the Total cost varies directly with the parameters T and θ, ie as T and θ increases or decreases, the total cost increases or decreases, whereas the Total cost varies inversely for the parameter γ, i.e as γ increases or decreases, the total cost decreases or increases accordingly. Also all three parameters are moderately sensitive.

Conclusion:

In this paper we have developed a two echelon supply chain model for time dependent deteriorating items for both manufacturer and retailer. It is often found that the demand of a product varies with time t and hence time dependent demand is taken fuzzy alongwith hexagonal fuzzy. In order to keep balance between production and demand, production is taken to be demand dependent. Under these conditions, we have determined the cost of total supply chain, optimal production time for the manufacturer and also the time at which the inventory becomes zero for the retailer under crisp, fuzzy and hexagonal fuzzy environment.

Bibliography

[1] L.A.Zadeh, "Infomation and Control," *Fuzzy Sets*, vol. 8, pp. 338-356, 1965.

[2] H. C. Chang, "An analysis of production inventory models with deteriorating items in a two echelom supply chain.," *Appl Math Model*, vol. 38, pp. 1187-1191, 2014.

[3] S. Tayal, S. R. Singh, R. Sharma and A. Chauhan, "Two echelon supply chain model for deterioration items with effective investment in preservation technology.," *Int J Math Oper Res*, vol. 6, p. 84, 2014.

[4] S. Saha and T. Chakrabarti, "A Fuzzy inventory model for deteriorating items with linear price dependent demand in a supply chain," *International Journal of Fuzzy Mathematical Archive*, vol. 13, no. 1, pp. 59-67, 2017.

[5] J. D. Sen and T. Chakrabarti, "An Industrial Production Inventory model with deterioration under neutrosophic fuzzy optimization," *Neuroquantology*, vol. 20, no. 19, pp. 4399-4406, 2022.

[6] E. Teimoury and S. M. Kazemi, "An integrated," *An Integrated Pricing and Inventory Model for Deteriorating Products in a two stage supply chain under Replacement and Shortage*, vol. 24, pp. 342 - 354, 2017.

[7] A. K. Bhunia, ,. C. K. Jaggi, A. Sharma and R. Sharma, "A two-warehouse inventory model for deteriorating items under permissible delay in payment with partial backlogging.," *Applied Mathematics and Computers*, vol. 232, pp. 1125 - 1137, 2014.

[8] J. D. Sen and T. Chakrabarti, "An EOQ model for healthcare industries with exponential demand pattern and time dependent delayed deterioration under fuzzy and neutrosophic environment.," *Neuroquantology,* vol. 20, no. 12, pp. 4080-4086, 2022.

[9] T. Chakrabarti and K. Chaudhury, "An EOQ model deteriorating items with a inear trend in demand and shortages in all cycles.," *International Journal of Production Economics,* vol. 49, no. 3, pp. 205-213, 1997.

[10] Y. W. Zhou, J. Min and S. K. Goyal, "A Supply-chain coordination under an inventory-level-dependent demand rate.," *International Journal of Prouction Economics,* vol. 113, pp. 518-527, 2008.

[11] S. Shee and T. Chakrabarti, "Fuzzy inventory Model for deteriorating items in a supply chain system with time dependent demand rate," *Soft Computing in Inventory Management Singapore,* pp. 69-82, 2021.

[12] C. K. Jaggi and A. Kauser, "A Retailer's Ordering Policy in a Supply Chain when Demand is Price and Credit Period Dependent.," *International Journal of Strategic Decision Sciences,* vol. 2, pp. 61-74, 2011.

[13] S. Saha and T. Chakrabarti, "A Two-Echelon Supply Chain Model for Deteriorating Product with Time-Dependent Demand, Demand-Dependent Production Rate and Shortage," *IOSR Journal of Engineering (IOSRJEN),* vol. 8, no. 11, pp. 33-38, 2018.

[14] J. D. Sen and T. Chakrabarti, "An EOQ model for time dependent deteriorating items with octagonal fuzzy Quadratic demand," *Mathematical Statiscian and Enginering Application,* vol. 72, no. 1, pp. 2271-2278, 2023.

[15] N. H. Shah, A. S. Gor and H. M. Wee, "An Integrated Approach for Optimal Unit Price and Credit Period for Deteriorating Inventory System when the Buyer's Demand is Price Sensitive," *Amer J Math Management Science,* vol. 30, pp. 317-330, 2010.

[16] J. D. Sen and T. Chakrabarti, "An EPQ model for products with two parameter Weibull distribution deterioration with fuzzy and decagonal fuzzy demand," *Journal of Survey in Fisheries Sciences,* vol. 10, no. 2, pp. 1565-1570, 2023.

[17] J. J. Liao, K. J. Chung and K. N. Huang, "A deterministic inventory model for deteriorating items with two warehouses and trade credit in a supply chain system," *International Journal of Production Economics,* vol. 146, pp. 557-565, 2013.

[18] X. Wang, W. Tang and R. Zhao, "Fuzzy economic order quantity inventory models without backordering," *Tsinghua Sci Technol,* vol. 12(1), p. 91, 2007.